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A Short Course On Differentiable Manifolds

Siddharta Sen
Trinity College, Dublin



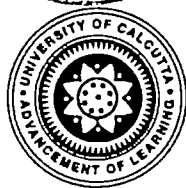
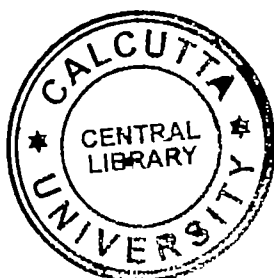
**UNIVERSITY OF CALCUTTA
2011**

Lectures given to M.Sc. students of the Department of Applied Mathematics, Calcutta University,
under UGC Special Assistance Scheme (II).

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40

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Preface

These lecture notes are based on a short introductory course on differential manifolds given to the M. Sc. students of the Department of Applied Mathematics of Calcutta University under a UGC programme. The topics covered were standard. The aim of the course was to present the standard topics with motivational remarks and calculational details that would be helpful for the students. Proofs of theorems were usually not provided. It is hoped that this course would help students to acquire the minimum background necessary for them to use the powerful tools and methods of manifold theory elsewhere. A significant omission due to lack of time is a discussion on fiber bundles. Unfortunately the lecture notes do not provide sufficient number of examples to properly illustrate the power and elegance of the tools introduced. For this, the student is urged to consult the references listed at the end of the book.

I would like to thank Prof. Bijan Bagchi, Head of the Department of Applied Mathematics, Calcutta University for organizing my visit and making my stay pleasant. I would like to thank Sayan Chakrabarti, research scholar of Theory Division of Saha Institute of Nuclear Physics, for taking notes of the course and for his great help in producing these lecture notes.

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Calcutta,
December 2008.

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Chapter 1

Introduction

1.1 Introduction

1.1.1 Motivational remarks

A differential manifold is a space that is locally Euclidean on which the operations of calculus can be carried out. A simple example is the surface of a sphere. The sphere is curved, however, a neighbourhood of a point on the sphere can be approximated by the tangent plane at that point which is a two dimensional Euclidean space. Furthermore the surface of the sphere can be covered by a finite number of neighbourhoods. Thus the sphere can be approximated by a collection of its tangent planes, i.e. by a collection of Euclidean spaces. We know how to carry out differentiations and integrations in Euclidean space. This suggests a way to introduce these operations on the surface of the sphere simply by using the local neighbourhoods on the sphere and maps from these local neighbourhoods to Euclidean space. The procedure described can be extended to more general spaces which have a local neighbourhood description and have the invertible maps from these neighbourhoods to the Euclidean space. The invertible maps from the general space $\mathcal{M}$ to Euclidean space provide Euclidean coordinates for the points on $\mathcal{M}$. Since these maps are invertible, they can be used to “pull back” the operations of calculus from Euclidean space to $\mathcal{M}$. For such a procedure to work, we need to have a precise understanding of when a space can be described in terms of a collection of neighbourhoods and invertible maps. This is the subject matter of topology. Note that in the procedure described it will very often be the case that a given point in the space will belong to more than one neighbourhood. It may lie, for instance, in a region which is contained in the intersection of two neighbourhoods. A consequence of this is that the point will have more than one Euclidean space coordinates

description since each neighbourhood will have its own map into Euclidean space. It is natural to interpret these different coordinate descriptions as coordinate changes. By using such a procedure, which we will describe in greater details shortly, the operations of calculus can be defined on the space of interest. If this can be done, our space is a differential manifold. The procedure sketched is of interest in mathematics as it allows the properties of general spaces to be explored using the operations of calculus. The procedure is also of interest in applied mathematics and physics where it can be used to physical systems of interest that take place in more general spaces, for instance, weather patterns on the surface of the earth or general dynamical systems. In both uses, it is necessary to introduce additional structures such as the metric tensor or the electromagnetic field tensor in order to properly capture properties of the system of interest. In order to introduce these structures in a satisfactory way we will need ideas from multilinear algebra.

1.2 Recap of some general ideas

We proceed to introduce the key ideas of topology and multilinear algebra that we need. Ideas from topology will clarify how a space can be describable by neighbourhoods which are locally Euclidean, while ideas from multilinear algebra will provide a general framework for treating concepts such as a metric tensor or electromagnetic field tensor in a uniform way.

1.2.1 Continuity

A key concept needed to start our discussions of differential manifolds is that of continuity. Continuity is the starting point for calculus. The mathematical discipline which studies continuous maps and continuity in a general setting is topology. We start by recalling the standard definition of continuity of a real valued function with variables in one dimensional Euclidean space (the real line). We then proceed to generalize the idea to a formulation of the concept of continuity which will be the starting point for introducing the notion of a topological space. We have the following

- f is a map $f : \mathbb{R} \rightarrow \mathbb{R}$ which maps $x \rightarrow f(x)$, $y \rightarrow f(y)$ then $f(x)$ is **continuous** at a point x if

$$|f(x) - f(y)| < \epsilon \Rightarrow |x - y| < \delta, \quad \epsilon, \delta > 0 \quad (1.1)$$

- Now we generalize this one dimensional case to $\mathbb{R}^n$

$$|f(\vec{x}) - f(\vec{y})| < \epsilon \Rightarrow |\vec{x} - \vec{y}| < \delta, \quad \epsilon, \delta > 0,$$

$$\text{where } |\vec{x} - \vec{y}| = \sqrt{\sum_{i=1}^n (x_i - y_i)^2} \quad (1.2)$$

Further generalizations are possible. To see this we introduce the idea of a metric space.

1.2.2 Metric Space

A metric space is a set S with a metric on it. The **metric** associates with any pair of elements (points) of S a distance.

♣ **Definition:** A **metric space** is a pair (S, d) , where S is a set and d is a metric on S (more precisely the distance function on S), such that $\forall x, y, z \in S$ we have

- d is real valued, finite and non negative,
- $d(x, y) \geq 0$,
- $d(x, y) = d(y, x)$,
- $d(x, y) + d(y, z) \geq d(x, z)$. (Triangle inequality).

♠ **Few examples:**

1. **Real line $\mathbb{R}$:** This is the set of all real numbers, taken with the usual metric defined by $d(x, y) = |x - y|$.
2. **Euclidean plane $\mathbb{R}^2$:** The metric space $\mathbb{R}^2$, called the Euclidean plane, is obtained if we take the set of ordered pairs of real numbers $x = (\xi_1, \xi_2)$, $y = (\eta_1, \eta_2)$, etc., and the Euclidean metric defined by

$$d(x, y) = \sqrt{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2} \quad (1.3)$$

We next introduce two key concepts:

1.2.3 Open and Closed sets

We start by defining the open and closed balls. The **open ball** in $\mathbb{R}^n$, of radius ϵ , centered at $\mathbf{a} \in \mathbb{R}^n$ is

$$B_{\mathbf{a}}(\epsilon) = \{\mathbf{x} \in \mathbb{R}^n \mid |\mathbf{x} - \mathbf{a}| < \epsilon\} \quad (1.4)$$

The **closed ball** is defined by

$$\bar{B}_{\mathbf{a}}(\epsilon) = \{\mathbf{x} \in \mathbb{R}^n \mid |\mathbf{x} - \mathbf{a}| \leq \epsilon\} \quad (1.5)$$

i.e., the closed ball is the open ball with its edges or boundaries included. Boundary points correspond to the set of points $|\mathbf{x} - \mathbf{a}| = \epsilon$. We can now define the open sets.

♣ **Definition:** A set U in $\mathbb{R}^n$ is called an **open set** if given any $\mathbf{a} \in U$ there is an open ball of some radius $r > 0$, centered at $\mathbf{a}$ that lies entirely in U . Clearly each $B_{\mathbf{b}}(\epsilon)$ is open if $\epsilon > 0$ (take $r = (\epsilon - |\mathbf{b} - \mathbf{a}|/2)$), whereas $\bar{B}_{\mathbf{b}}(\epsilon)$ is not open because of its boundary points. $\mathbb{R}^n$ itself is trivially open. A set F in $\mathbb{R}^n$ is called a **closed set** if its complement $\mathbb{R}^n - F$ is open.

It can be seen immediately that the union of any arbitrary collection of open sets in $\mathbb{R}^n$ is an open set and the intersection of any finite number of open sets in $\mathbb{R}^n$ is open.

Keeping these properties in mind we now proceed to a definition of continuity which does not depend on the metric. Why do we do this? There are a number of reasons that can be given. Mathematically such a step places the subject matter in the very general framework of topology (rubber sheet geometry), which is a sensible thing to do. At the level of application, introducing this general setting for continuity also makes sense. There are different problems in which different metrics might appear, so having a framework which is metric independent guarantees that the results established are all valid. There are also problems where a qualitative approach is required. In such problems the precise details of the metric may not be important. The way we get rid of the metric in defining continuity is to first reformulate the concept of continuity introduced using open sets. Open sets initially are defined in terms of the metric but will then be taken to be primitive elements of the system. We recall that continuity of a function $f : M \rightarrow N$ (M, N , two metric spaces) is defined as follows: For $f(x), f(y)$ elements of an open set in N (defined in terms of an ϵ i.e. $d(f(x), f(y)) < \epsilon$), then points x, y must be in an open set $|\mathbf{x} - \mathbf{y}| < \delta$ in M . That is, for f to be continuous, the inverse map of an open set in N is an open set in M . This is the key requirement. To study continuous maps we thus need open sets.

We are now ready to introduce a general space called a topological space in which the concept of continuity can be defined without using the idea of a metric. The basic objects of a topological space will be a prescribed collection of open sets. Continuity of a function will follow if the inverse map of an open set (from the prescribed collection) is an open set (in the prescribed collection). Independence of the idea of a metric is achieved by thinking of the open sets as primitive objects which are to be specified for a given space.

♣ **Definition:** We can thus define a **topological space** as a space with a given collection of open sets U_α which have the following properties

- X, ϕ (the null set) $\in \{U_\alpha\}$,
- Arbitrary union of $U_\alpha \in U_\alpha$,
- Finite intersection of $U_\alpha \in U_\alpha$,

A space X with such a structure is a **topological space**. Note that the primitive notion of open sets introduced does not use the idea of distance. They are simply the way the topological space X is set up. The local description of X is provided by the open sets U_α , from this informations global properties of X are studied in topology.

♠ **Few examples:**

1. X, ϕ are the only open sets.
2. All sub sets of X are open sets of X .

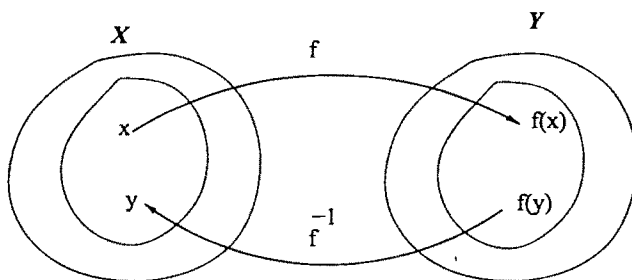


Figure 1.1: A map f from a topological space X to another topological space Y . f is continuous if f^{-1} maps every open sets of Y into open sets of X

If $f : X \rightarrow Y$ is a map of a topological space X into another topological space Y , we can now simply say that f is **continuous** if f^{-1} maps open sets of Y into open sets of X .

Now that we have introduced the concept of a topological space it is necessary to know how to decide if two topological spaces X and Y are the “same” or “different”. X and Y are the same (X **homeomorphic** to Y) if continuous maps $f : X \rightarrow Y$ and $g : Y \rightarrow X$ exist such that

$$\begin{aligned} f \circ g &= 1_X : \text{identity map on } X \\ g \circ f &= 1_Y : \text{identity map on } Y \end{aligned} \quad (1.6)$$

The condition of equality introduced, in words, say that X is homeomorphic to Y (“ $X = Y$ ”) if X can be mapped in a continuous one to one way to Y by f and similarly Y can be mapped in a continuous one to one way to X by g . Although the way to decide if spaces X and Y are homeomorphic is clear, an algorithmic method for implementing the procedure does not exist for arbitrary spaces.

Let us discuss about the subspace topological space now, which is an important concept

- The useful concept of a subspace S of a topological space X can be easily introduced. To do this we need to give a procedure for determining the open sets of S required for giving it a topology. The open sets of the subspace can be constructed from those of X by simply taking the intersection of S with the open sets which define the topology of X . This defines what is known as the relative topology on S . Thus the open sets V_α of S are constructed from the open sets U_α which define the topology of X as

$$V_\alpha = U_\alpha \cap S \quad (1.7)$$

- The subspace topological space is a way of constructing a new topological space from a given topological space X . Another standard way of constructing a new topological space is by starting from a pair of topological spaces X , Y , and taking their tensor product. The open sets which define the topology of this space are simply ordered pairs of the basis open sets of X and Y . The concepts of tensor product and basis will be defined shortly.

We collect a few essential facts regarding the topological spaces that we need

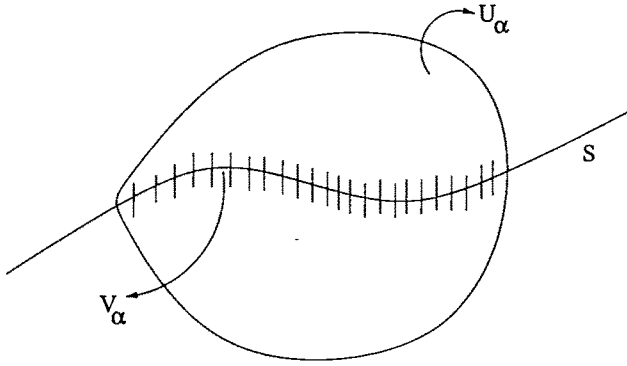


Figure 1.2: Open sets of relative topology

1. ♣ **Definition:** A space $\hat{X}$ has a **cover** if

$$\hat{X} = \bigcup_{\alpha} U_{\alpha}, \quad (1.8)$$

where U_{α} are open sets.

2. ♣ **Definition:** A space $\hat{X}$ is **compact** if a cover of $\hat{X}$ can be replaced by a cover with only a finite number of terms.
3. ♣ **Definition:** A topological space X is **paracompact** if for every open covering U of X there exists a locally finite open covering V . Simply the space X need not have a global finite cover but can have a finite cover for any local neighbourhood. An example of such a space is Euclidean space.

The property of paracompactness is crucial for constructing theory of integration on a manifold, as we will see, when we discuss Stoke's theorem. It is also a property that allows a metric (length scale) to be introduced on a space.

4. ♣ **Definition:** A space X is **Hausdorff** if given $x, y \in X$ there are open sets U_{α}, U_{β} such that $x \in U_{\alpha}; y \in U_{\beta}$ and $U_{\alpha} \cap U_{\beta} = \phi$, i.e. any two points can be enclosed in disjoint open sets. The Hausdorff property is crucial for sequences to converge in a space.
5. ♣ **Definition:** There are different notions of **connectedness**. The most primitive notion is: X is **connected** if X can not be written as $X = A \cup B$ where $A \cap B = \phi$ but $A, B \neq \phi$. i.e. a topological space

X is connected if the only sets which are both open and closed are ϕ and X .

6. $\diamond$ **Theorem:** A topological space X is connected if and only if it is not the union of two disjoint non-empty open sets.

1.2.4 Basis for a topological space

For checking if a given space is a topological space, the idea of a basis $\mathcal{B}$ for a topological space can be useful.

♣ Definition: A collection of open sets form a basis $\mathcal{B}$ for a topological space X if each open set belonging to X can be expressed as a union of members of $\mathcal{B}$.

We also note that a subset of open sets of X can form a basis if $X = \bigcup_b B_b$, and if the intersection of any two open sets of X can be written as the union of members of $\mathcal{B}$ (see fig (1.3)).

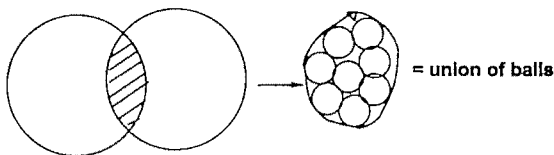


Figure 1.3: Union of small balls

Remarks: It is clear that any space with a metric is a topological space. We simply choose open sets $\mathcal{B}_{x_0}^{(a)} = \{x, d(x, x_0) < a\}$, i.e balls centered at x_0 of “size” a as the basis $\mathcal{B}$

1.3 Differentiable Manifolds ($\mathcal{M}$)

We are now in a position to give a proper definition of a differentiable (differential) manifold $\mathcal{M}$. Unless otherwise stated, by a manifold we will mean a differentiable (differential) manifold. There are two parts in the definition. The first part requires a differential manifold of dimension n to be a topological space that is Hausdorff and is locally homeomorphic to n dimensional Euclidean space. This is a local condition. The second part requires the local homeomorphism to satisfy certain smoothness conditions. This condition specifies the way the local information about the manifold is glued together. We briefly comment on the requirements listed. The requirement that $\mathcal{M}$ be a topological space means that $\mathcal{M}$ can be described by a collection of open

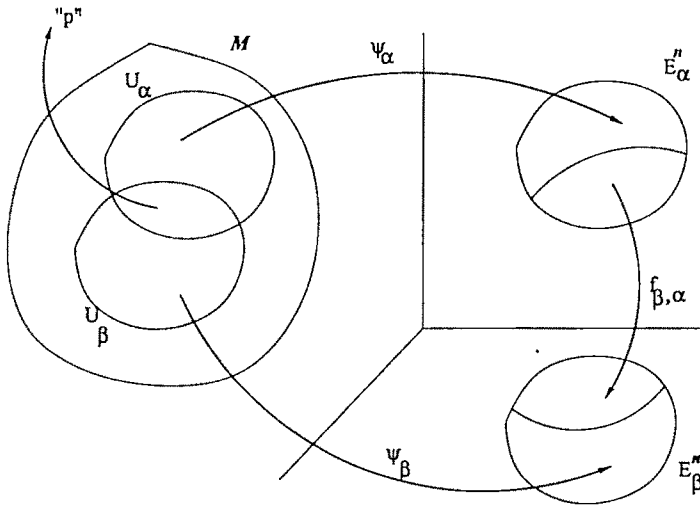


Figure 1.4:

sets. These open sets provide the local neighbourhoods which we will use to implement the requirement that $\mathcal{M}$ is locally Euclidean through invertible maps. The requirement that $\mathcal{M}$ is Hausdorff means that if a convergence sequence is defined on $\mathcal{M}$, it will have a unique limit. The requirement that $\mathcal{M}$ is locally homeomorphic to Euclidean space means that there are one to one invertible maps from the neighbourhoods (open sets) of $\mathcal{M}$ to Euclidean space. These maps provide Euclidean coordinates for points on $\mathcal{M}$. We now turn to second requirement regarding the smoothness for the local maps introduced. To do this we consider a point x in $\mathcal{M}$ that belongs to two overlapping neighbourhoods of $\mathcal{M}$ — U_α and U_β . If ψ_α and ψ_β are homeomorphisms from U_α and U_β to Euclidean space $\mathbb{R}^n$ then using ψ_α and ψ_β a map $f_{\beta\alpha} = \psi_\beta \circ \psi_\alpha^{-1}$ between points in Euclidean space can be constructed (see fig(1.4)). These maps from overlap regions of neighbourhoods are called the transition functions of $\mathcal{M}$. The transition functions describe how information contained in different neighbourhoods are to be glued together in a consistent way. We require these transition functions to have infinite number of derivatives. They are said to be C^∞ functions of their Euclidean coordinates. It is possible to consider spaces where the transition functions are only continuous or have only a continuous k derivatives. In these cases we have continuous C^0 or continuous C^k manifolds. For differential manifolds the requirement is for the transition functions to be infinitely differentiable. Such functions are sometimes said to be smooth functions. We summarise our remarks in the form of a definition:

♣ **Definition:** $\mathcal{M}$ is a differential manifold if:

1. $\mathcal{M}$ is a Hausdorff topological space,
2. $\mathcal{M}$ has a family of pairs of local neighbourhoods and maps $\{(U_\alpha, \psi_\alpha)\}$,
3. The neighbourhoods U_α are a family of open sets which cover $\mathcal{M}$: $\bigcup_\alpha U_\alpha = \mathcal{M}$. The ψ_α are homeomorphisms from U_α to open subsets E_α of $\mathbb{R}^n$,
4. Given overlapping neighbourhoods U_α and U_β , the map $\psi_\beta \circ \psi_\alpha^{-1}$ from the open subset $\psi_\alpha(U_\alpha \cap U_\beta)$ of $\mathbb{R}^n$ to the open subset $\psi_\beta(U_\alpha \cap U_\beta)$ of $\mathbb{R}^n$ is infinitely differentiable written C^∞ .

The family of neighbourhoods and maps $\{(U_\alpha, \psi_\alpha)\}$ introduced above is called an atlas. The individual members (U_α, ψ_α) of the family are called charts.

The maps ψ_α provide Euclidean coordinates for the neighbourhood U_α . For instance a point p contained in $U_\alpha \cap U_\beta$ in $\mathcal{M}$ will have the coordinates $\psi_\alpha(p) = (x_\alpha^1, \dots, x_\alpha^n)$ or $\psi_\beta(p) = (x_\beta^1, \dots, x_\beta^n)$. These two sets of coordinates are explicitly related by

$$x_\beta^i = f_{\beta\alpha}^i(x_\alpha^1, \dots, x_\alpha^n) \quad (1.9)$$

The smoothness requirement for transition functions means that the functions $f_{\beta\alpha}^i$ are infinitely differentiable functions of $(x_\alpha^1, \dots, x_\alpha^n)$.

1.3.1 The example of a circle (S^1):

A general manifolds cannot be covered by a single chart. If that was the case then all manifolds would be compact. Euclidean space, itself, is not compact and requires an infinite number of charts for its definition. However, Euclidean space can be covered by a countable infinite number of charts. Local charts can be defined in terms of spherical balls centred at points in Euclidean space. If the location of these centres is taken to be a collection of rational numbers then the number of balls will be countably infinite. The key point is that the rationals are dense in the reals. Consider now a simple example of a manifold, namely a circle S^1 . There are many different ways of describing S^1 . Algebraically we can think of it as points (x, y) such that $x^2 + y^2 = 1$ or as points on the real line where points x and $x+L$ are identified or as the space defined by the set of equations: $x^2 + y^2 + z^2 = 1$ and $z = 0$. Each one of these descriptions has a generalization useful for studying more general spaces. The method of identification stated is a simple example of what is known as the quotient construction. Describing S^1 in terms of two

algebraic equations is a simple example of the general idea of an algebraic variety. We now proceed to define S^1 as a differential manifold. To do this we need to introduce charts that cover S^1 and maps from these charts which give coordinates. The collection of charts, thus provides a coordinate system. Let us examine the situation. A minimum of two coordinate charts are required for the circle. These are shown in the figure (1.5)

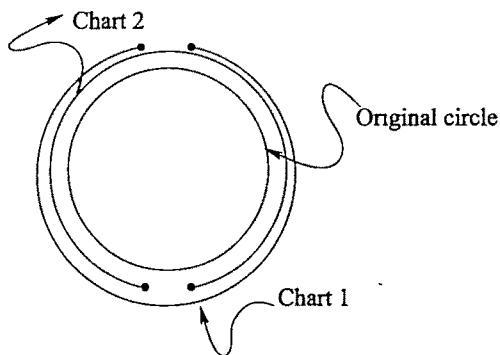


Figure 1.5: Two different charts covering the circle S^1

Let us give some details. A chart must relate a region of S^1 to Euclidean space. We take S^1 to be the set of points on $\mathbb{R}^2$ defined by $S^1 : \{(y, x) \mid x^2 + y^2 = 1\}$. A chart maps the open set U_1 , defined to be the circle minus the point $\theta = 0$, to the Euclidean line (the x axis via a stereographic projection as illustrated in the figure (1.6)).

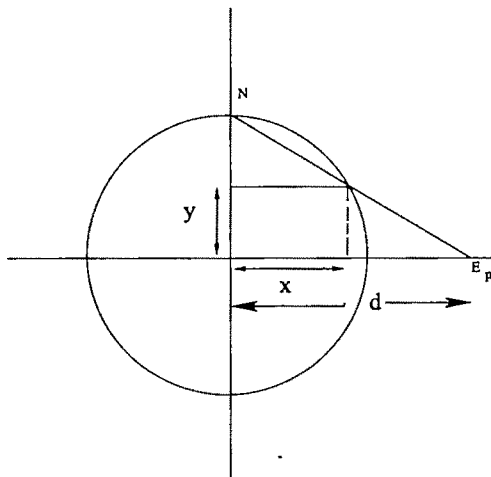


Figure 1.6: Defining a stereographic coordinate chart on S^1

We draw a straight line from the point $\theta = 0$ to the line $y = 0$, and get a point on the x axis, namely, the point E_p in the figure (1.6). Explicitly we have:

$$\begin{aligned} \frac{x}{d} &= \frac{1-y}{1}, \\ d &= \frac{x}{1-y} \end{aligned} \quad (1.10)$$

The resulting coordinates cover the entire circle except the point $\theta = 0$. Similarly one can think of the chart which covers the circle except the point $\theta = 2\pi$. These two charts thus cover all points of the circle.

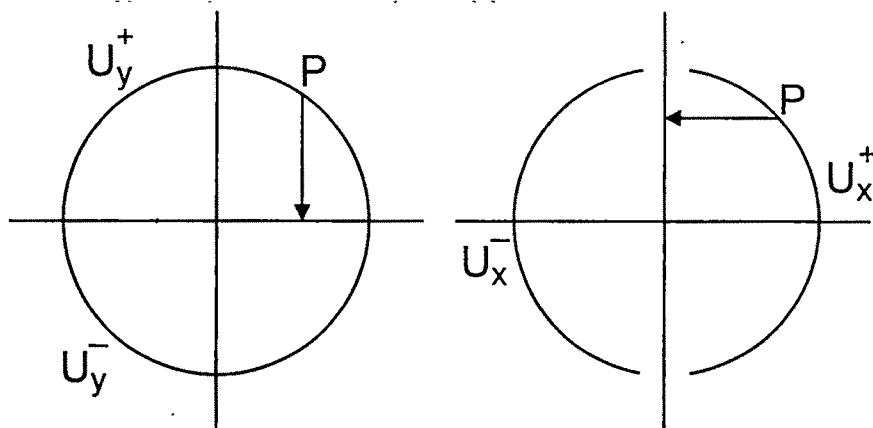


Figure 1.7: A set of different charts for the circle S^1 .

Other descriptions of S^1 with different homeomorphisms to Euclidean space are also possible. We give an example where S^1 is covered by four charts. Each one of these charts correspond to an open half circle. The way points from S^1 are mapped to Euclidean space from these charts can be understood from the figure (1.7) where a mapping from two charts to Euclidean space is shown by an arrow. In the figure a point p , which belongs to two charts is considered. A map between these Euclidean points is constructed by simply starting from one of these points using the inverse homeomorphism to reach $p \in S^1$ and then using a homeomorphism from the second chart to reach the other Euclidean point.

§ **Problem:** Generalize the example of S^1 to show that S^2 is a manifold.

1.3.2 Orientation of a Manifold

A two dimensional sphere in three dimensions has an inside and outside. It is two sided. This is also true for a disc. In both these cases we have a two dimensional surface placed in three dimensions where the normal to the surface at a point can be taken to point in any one of two opposite directions. A choice regarding which of these two directions is chosen as the direction of the normal has to be made. These two choices represent two different orientations that can be introduced on the surface. The orientation, once chosen, at a point on the surface, can be used to orient near points by infinitesimal coordinate changes. Such coordinate changes can be represented by a matrix. The orientation will be the same if the determinant of this matrix is positive. This simply means that if a right handed screw motion fixes the direction of the normal to the surface, then this property remains unchanged at the nearby point. If by this procedure all points on the surface can be given the same orientation then the surface is said to be orientable. In our trivial example, this is true. However there are surfaces which are not orientable. An example of such a surface is a Möbius strip. One can move along the surface to find that the direction of the normal to the surface is reversed (see figure 1.8).

Mathematically the key object for establishing orientability is the sign of the Jacobian determinant. Two different bases $\{v_i\}$ and $\{w_i\}$ related by a matrix A : $v_i = A^j_i w_j$, have the same orientation if $\det A > 0$. A manifold is orientable if for all such coordinate changes the determinant is always positive.

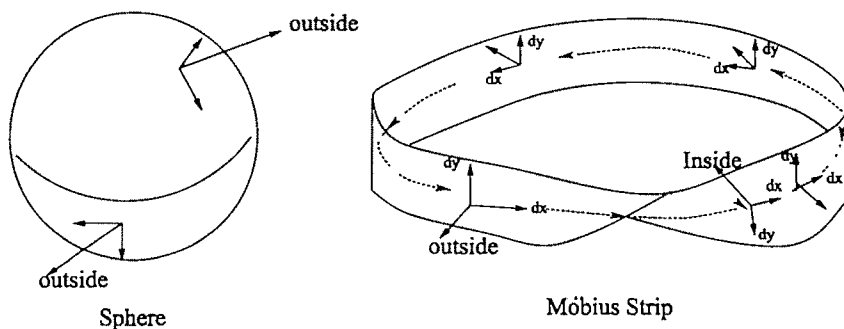


Figure 1.8: Example of an orientable and a non-orientable surface

Analogously one can consider a manifold which is covered by charts (V_α, U_α) . There the coordinates are given by homeomorphisms ψ_α and one has to study the coordinate change $\psi_\beta \cdot \psi_\alpha^{-1}$. Then the manifold is orientable if there is an atlas such that $\det(\psi_\beta \cdot \psi_\alpha^{-1}) > 0$, $\forall V_\alpha, V_\beta$ with $V_\alpha \cap V_\beta \neq \emptyset$.

A Möbius strip can be constructed by taking a strip of paper and gluing the opposite edges together after a twist through π . Because of the twist, the surface constructed has only one side.

1.3.3 Maps between Manifolds

So far we have considered a single differentiable manifold. It is often necessary to introduce maps from the manifold to itself or maps between different manifolds. Maps of interest are those that are either injective (one to one) or surjective (onto). A surjective (onto) map implies that the image of the map covers every point of the target manifold. For an injective map this is not necessarily true. Consider two differentiable manifolds $\mathcal{M}$ and $\mathcal{N}$. Let $f : \mathcal{M} \rightarrow \mathcal{N}$. We will consider below three different cases:

1. $\dim \mathcal{M} = \dim \mathcal{N}$
2. $\dim \mathcal{M} < \dim \mathcal{N}$
3. $\dim \mathcal{M} > \dim \mathcal{N}$

We consider each of these cases separately.

$\dim \mathcal{M} = \dim \mathcal{N}$

Consider a point p in an open neighborhood V_α of a manifold $\mathcal{M}$ and a map from the manifold $\mathcal{M}$ to another manifold $\mathcal{N}$ such that $f : p \rightarrow f(p)$ then a linear map from the tangent space at p of the manifold ($\mathcal{M}$) to the tangent space at $f(p)$ of the manifold ($\mathcal{N}$) can be defined as follows:

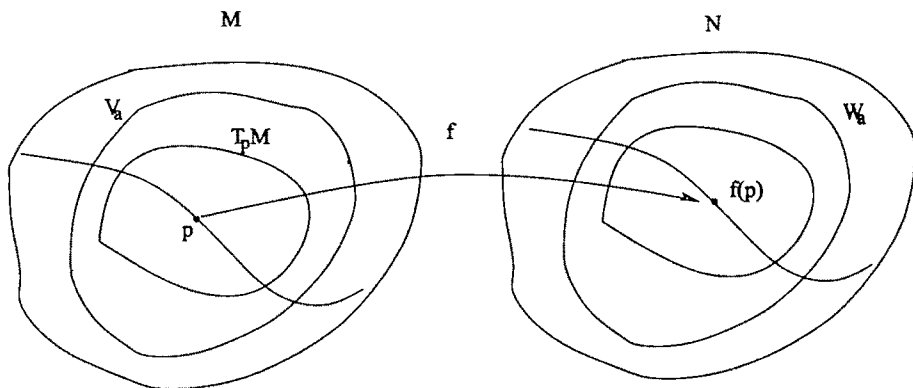


Figure 1.9: Tangent space on a Manifold at a point p

$$(df_h)_p = \lim_{t \rightarrow 0} \frac{f(p+th) - f(p)}{t} \quad (1.11)$$

If f is a smooth invertible one to one onto map, then f is said to be a **diffeomorphism**. We have the following remarkable theorem:

◇ **Theorem:** Suppose that $f : X \rightarrow Y$ is a smooth map whose derivative $(df_h)_p$ has a non-zero determinant, then f is a local diffeomorphism.

The remarkable feature of this theorem, known as the **Inverse Function Theorem**, is that the complicated non-linear property of a map being a diffeomorphism is decided locally by simply checking that one number, namely the determinant of $(df_h)_p$, is non-zero.

1.3.4 Push forward & Pull back maps

A map from $\mathcal{M} \rightarrow \mathcal{N}$ corresponds to $x \rightarrow y(x)$ where $x \in \mathcal{M}$ and $y \in \mathcal{N}$. This leads to $\frac{\partial}{\partial x} \rightarrow \frac{dy}{dx} \frac{\partial}{\partial y}$ which means that the mapping $y(x)$ can be used to change $\frac{\partial}{\partial x}$ to $\frac{\partial}{\partial y}$. These are examples of a **push forward** map. The map can be used to push forward $\frac{\partial}{\partial x}$ to $\frac{\partial}{\partial y}$. The situation for a differential dy is different. The function $y(x)$ can be used to transform dx to dy by $\frac{dy}{dx} dx \rightarrow dy$. This is an example of a **pull back**.

$\dim \mathcal{M} < \dim \mathcal{N}$

Let, $\dim \mathcal{M} = k$ and $\dim \mathcal{N} = n$ and $n > k$.

◇ **Theorem:** $f : \mathcal{M} \rightarrow \mathcal{N}$ is an **immersion** if locally one can write

$$f(x_1, x_2, \dots, x_k) = (x_1, x_2, \dots, x_k, 0, 0, \dots, 0). \quad (1.12)$$

Locally the map is one to one (injective).

$\dim \mathcal{M} > \dim \mathcal{N}$

◇ **Theorem:** $f : \mathcal{M} \rightarrow \mathcal{N}$ is a **submersion** if locally one can write

$$f(x_1, x_2, \dots, x_n, \dots, x_k) = (x_1, \dots, x_n). \quad (1.13)$$

Locally the map is onto (surjective). Although we will not discuss this case any more we remark that by studying the inverse map, from the lower dimensional manifold, to the higher dimensional manifold considerable information regarding the higher dimensional manifold can be obtained.

Submanifolds

There is a natural notion of a submanifold of a given manifold. To define a submanifold we need to give the coordinate charts associated with it. These charts can be constructed by taking the intersection of the submanifold with the coordinate charts of the original manifold (see figure (1.10)). Let f be a map from $\mathcal{M}$ to $\mathcal{N}$ and $\dim \mathcal{M} < \dim \mathcal{N}$.

Then if a coordinate neighborhood of $\mathcal{N}$ can be written as $U_{\mathcal{N}}^{\alpha} = \mathcal{N} \cap U_{\mathcal{M}}^{\alpha}$, where $U_{\mathcal{M}}^{\alpha}$ is an open neighborhood of $\mathcal{M}$, then $\mathcal{N}$ is a **submanifold** of $\mathcal{M}$.

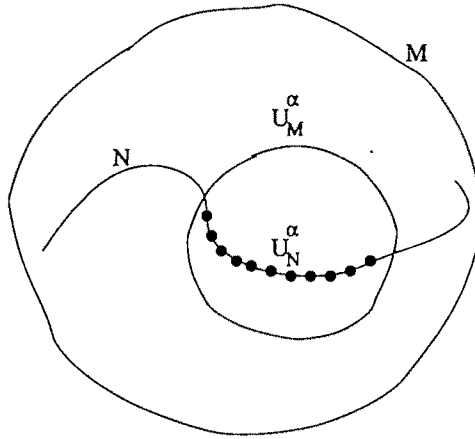


Figure 1.10: Submanifold $\mathcal{N}$ of a manifold $\mathcal{M}$

1.4 Some basic statements of Multilinear Algebra

We start by quickly summarizing some facts about finite dimensional vector spaces. If V is a finite dimensional vector space of dimension k , it means that

- $v_1 \in V \Rightarrow \lambda_1 v_1 \in V$, where λ_1 is a scalar. If λ_1 is real (complex), the vector space is said to be a real (complex) vector space.
- $v_1, v_2 \in V \Rightarrow \lambda_1 v_1 + \lambda_2 v_2 \in V$, where λ_1, λ_2 are scalars.

♣ **Definition:** A collection of vectors $v_1, \dots, v_n \in V$ are said to be linearly independent if $\sum_{i=1}^n \lambda_i v_i = 0 \Rightarrow \forall \lambda_i = 0$

♣ **Definition:** The maximum number of linearly independent vectors in V is the dimension of V .

◇ **Theorem:** Two vector spaces V and W are equal (isomorphic) if $\dim V = \dim W$.

◇ **Theorem:** For a vector space V of dimension k , it is possible to select a set of k linearly independent basis vectors $e^1, \dots, e^k$ such that any $v \in V$ can be written as $v = \sum_{i=1}^k \lambda_i e^i$.

♣ **Definition:** Associated with a vector space V of dimension k , there is a dual vector space V^* of the same dimension. Elements $v^* \in V^*$ are defined as follows: given $v \in V$ and $v^* \in V^*$, $v^*(v)$ is a linear scalar function of v . We write $\langle v^*, v \rangle = v^*(v)$.

Note that if T is a linear map from a vector space V to W , i.e. $T : V \rightarrow W$, then there is an induced map $T^* : W^* \rightarrow V^*$ (the direction of the map is reversed). To see this we note $\langle w^*, Tv \rangle \stackrel{\text{def}}{=} \langle T^* w^*, v \rangle = \langle v^*, v \rangle$. Thus $T^* : W^* \rightarrow V^*$.

We now turn to considerations involving a manifold. The tangent space at a point p of a manifold has the same dimension (k) as the manifold and gives a linear approximation to the manifold in the neighborhood of the point p . It is a vector space. Let us call this vector space V . We now extend the idea of using linear functions on a vector space V to generate a new vector space, (its dual V^*) by considering real valued functions T on the Cartesian product

$$T = \underbrace{V \times V \times \dots \times V}_r \times \underbrace{V^* \times V^* \times \dots \times V^*}_s \rightarrow \mathbb{R} \quad (1.14)$$

which are separately linear in each variable (multilinear). This real valued function is called a tensor of type (r, s) and is written as $T(v_1, \dots, v_r, v_1^*, \dots, v_s^*)$. Tensors can be multiplied. If T is a tensor of type (r_1, s_1) and S is a tensor of type (r_2, s_2) , then the tensor product of T and S written as $T \otimes S$ gives a tensor of type $(r_1 + r_2, s_1 + s_2)$. It is defined by

$$\begin{aligned} & (T \otimes S)(v_1, \dots, v_{r_1}, \dots, v_{r_1+r_2}, \dots, v_1^*, \dots, v_{s_1}^*, \dots, v_{s_1+s_2}^*) \\ &= T(v_1, \dots, v_{r_1}, v_1^*, \dots, v_{s_1}^*) S(v_{r_1+1}, \dots, v_{r_1+r_2}, v_{s_1+1}^*, \dots, v_{s_1+s_2}^*). \end{aligned} \quad (1.15)$$

Note that $T \otimes S \neq S \otimes T$. However, using the definition given, it is easy to check that the tensor product rule is associative i.e. $T \otimes (S \otimes V) = (T \otimes S) \otimes V$

§ **Problem:** Prove that $T \otimes (S \otimes V) = (T \otimes S) \otimes V$.

Bilinear symmetric tensors of type $(0, 2)$ and completely anti-symmetric tensor of type $(0, p)$ are of particular interest. The symmetric tensor represents the metric tensor and provide generalization needed to discuss multidimensional integrals. The vector space of completely anti-symmetric p -linear functions on V form a vector space written $\Lambda^p(V^*)$. This is a vector space constructed from elements of the dual of V , namely V^* . For $\phi \in \Lambda^p(V^*)$, we have

$$\begin{aligned}\phi(v_1, \dots, v_i + w_i, \dots, v_p) &= \phi(v_1, \dots, v_i, \dots, v_p) \\ &+ \phi(v_1, \dots, w_i, \dots, v_p), \quad \forall i\end{aligned}$$

and

$$\begin{aligned}\phi(v_1, \dots, \lambda v_i, \dots, v_p) &= \lambda \phi(v_1, \dots, v_i, \dots, v_p), \quad \lambda : \text{Scalar}, \\ \phi(v_{\pi(1)}, \dots, v_{\pi(p)}) &= (-1)^\pi \phi(v_1, \dots, v_p),\end{aligned}$$

where π is an element of the permutation group S_p , which acts on the set $(1, \dots, p)$. Note that $(-1)^\pi = \pm 1$ for even/odd permutations.

By anti-symmetrizing the tensor product introduced, a new product called the **wedge product** can be defined which combines $\tau^p \in \Lambda^p(V^*)$ and $\mu^q \in \Lambda^q(V^*)$ to give an element written as $\tau^p \wedge \mu^q \in \Lambda^{p+q}(V^*)$. Explicitly we have

$$\begin{aligned}&\tau^p \wedge \mu^q(v_1, \dots, v_{p+q}) \\ &= \frac{1}{(p+q)!} \sum_{\pi \in S_{p+q}} (-1)^\pi \tau^p(v_{\pi(1)}, \dots, v_{\pi(p)}) \mu^q(v_{\pi(p+1)}, \dots, v_{\pi(p+q)})\end{aligned}$$

◇ **Theorem:** If $e_1, \dots, e_k$ is a basis for V^* which is k -dimensional then $e_{i_1} \wedge \dots \wedge e_{i_p}$ is a basis for $\Lambda^p(V^*)$, where $1 \leq i_1 \leq i_2 \leq \dots \leq i_p \leq k$.

To prove this result we need to show that if

$$\sum_{1 \leq i_1, \dots, i_p \leq k} \alpha^{i_1, \dots, i_p} e_{i_1} \wedge \dots \wedge e_{i_p} = 0 \quad (1.16)$$

then $\alpha^{i_1, \dots, i_p} = 0, \forall i_1, \dots, i_p, 1 \leq i_1, \dots, i_p \leq k$. Take the wedge product of the equation with $e_{j_1} \wedge \dots \wedge e_{j_{k-p}}$, then we have

$$0 = \sum \alpha^{i_1, \dots, i_p} e_{i_1} \wedge \dots \wedge e_{i_p} \wedge e_{j_1} \wedge \dots \wedge e_{j_{k-p}}. \quad (1.17)$$

But, $e_{i_1} \wedge \dots \wedge e_{i_p} \wedge e_{j_1} \wedge \dots \wedge e_{j_{k-p}} = 0$ if the set $(i_1, \dots, i_p)$ or $(j_1, \dots, j_{k-p})$ have any common element. Also the total number of factors in the wedge

product is k which is the dimension of V^* . Thus for any given set $(j_1, \dots, j_{k-p})$ there is exactly one set $(i_1, \dots, i_p)$ with no common element. Let us call this set $(\hat{i}_1, \dots, \hat{i}_p)$. Reordering the basis vectors we have $\alpha^{\hat{i}_1 \dots \hat{i}_p}(e_1 \wedge \dots \wedge e_k) = 0$, with the sign determined from the number of permutations needed to bring the basis vectors to the standard form. But $e_1 \wedge \dots \wedge e_k$ is proportional to the volume of V^* and is not zero. Thus $\alpha^{\hat{i}_1 \dots \hat{i}_p} = 0$. By varying $(j_1, \dots, j_{k-p})$, the result follows.

Exterior algebra structure

The space of wedge products form the exterior algebra, i.e. elements can be added and multiplied. For $\omega^p \in \Lambda^p(V)$ and $v^q \in \Lambda^q(V)$, the product $\omega^p \wedge v^q \in \Lambda^{p+q}(V)$. In terms of basis vectors

$$\omega^p \wedge v^q = (e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_p}) \wedge (e_{j_1} \wedge e_{j_2} \wedge \dots \wedge e_{j_q})$$

Interchanging ω and v is done by moving each of the factors of ω across the q factors of v . There are p such factors in ω , each one generates a factor $(-1)^q$. thus interchanging ω and v gives a sign factor of $(-1)^{pq}$.

If $(v_1, \dots, v_p) \in V$, the value of $e_{i_1} \wedge \dots \wedge e_{i_p}$ on these vectors is given by

$$(e_{i_1} \wedge \dots \wedge e_{i_p})(v_1, \dots, v_p) = \frac{1}{p!} \sum_{\pi \in S_p} (-1)^\pi e_{i_1}(v_{\pi(1)}) \dots e_{i_p}(v_{\pi(p)}). \quad (1.18)$$

The space $\mathcal{A}(V^*) = \sum_{j=0}^k \oplus \Lambda^j(V^*)$ with $\Lambda^0(V^*) = \mathbb{R}$ where multiplication between elements of $\mathcal{A}(V^*)$ is defined in terms of the wedge product of forms represents an exterior algebra. Elements of $\mathcal{A}(V^*)$ are called forms on V . Forms in $\Lambda^p(V^*)$ are said to be of degree p .

♠ Example:

Let us now consider the top form for a two dimensional space $\omega^{(2)} = dx \wedge dy$. We take $e_1 = dx$ and $e_2 = dy$. We will justify such an assignment later. We now verify that changing from Euclidean to polar coordinates will produce a top form which is related to the local area of the two dimensional space.

In polar coordinate $x = r \cos \theta$, $y = r \sin \theta$, hence

$$\begin{aligned} e_1 &= dx = dr \cos \theta - r \sin \theta d\theta \\ e_2 &= dy = dr \sin \theta + r \cos \theta d\theta \end{aligned}$$

So, we get

$$\begin{aligned}e_1 \wedge e_2 &= dx \wedge dy = (dr \cos \theta - r \sin \theta d\theta) \wedge (dr \sin \theta + r \cos \theta d\theta), \\&= (dr \wedge d\theta)r \cos^2 \theta - r \sin^2 \theta (d\theta \wedge dr); [\text{since } dr \wedge dr = d\theta \wedge d\theta = 0,] \\&= (dr \wedge d\theta)r(\cos^2 \theta + \sin^2 \theta), \\&= r(dr \wedge d\theta), \\&= e_r \wedge e_\theta,\end{aligned}\tag{1.19}$$

introducing $e_r = dr$ and $e_\theta = r d\theta$.

Chapter 2

Structures and tools

2.1 Introduction

Fundamental laws of physics as well as important equations in applied mathematics can be formulated in a compact way on a manifold by using the ideas of differential forms and vector fields. Such a formulation becomes effective after a number of useful operations are introduced. The operations are those of the Lie derivative, which is related to symmetries present in a system, the wedge product of forms, the exterior derivative operator, the interior product and the Hodge star operator. In this chapter these operations will be introduced. A few simple examples illustrating this approach are given in the next chapter.

2.2 Structure and tools on a manifold

In this section we introduce vector fields and differential forms. The vector field will be introduced as an operator which changes the coordinate at a point, while the differential form will be regarded as a function of a vector field. To motivate this point of view we consider the differential df associated with a real valued function f . In terms of coordinates

$$df(x^1, \dots, x^k) = \sum_{i=1}^k \frac{\partial f}{\partial x^i} dx^i \quad (2.1)$$

The geometrical meaning of df is clear. It represents a change in the value of the function due to an infinitesimal change in the coordinates. But, a coordinate change results, as we will show, from the action of a vector field on the coordinate functions. Furthermore, vector fields will be shown to form

a vector space. For infinitesimal coordinate changes df is a linear function of a vector field. df is thus an element of the space dual to the vector space of a vector field. This dual space is the one where differential forms will be defined. Thus, before we introduce differential forms we need to properly understand the way a vector field can be introduced as a differential operator.

2.2.1 Vector Field

Let us consider the way a function on the manifold changes due to infinitesimal changes in its coordinates as it moves along a curve $p(t)$. The infinitesimal change of coordinates can be described in a geometrical way as due to the action of a differential operator to be called a vector field on the coordinate functions. An infinitesimal change of the coordinates on a curve can also be used to construct a basic geometrical object namely the tangent vector to the curve. We can bring these two notions together by noting that each coordinate is a function. Explicitly we note that the tangent vector to the curve $p(t)$ is given by its coordinate functions

$$\frac{d}{dt}x^i(p(t)) \equiv v^i. \quad (2.2)$$

On the other hand the change of a general function of the coordinates $f(p(t)) : \mathcal{M} \rightarrow \mathbb{R}^k$, as it moves along the curve $p(t)$ can be written as:

$$\frac{d}{dt}f(p(t)), \quad (2.3)$$

which in terms of the local coordinates is equal to

$$\sum_{i=1}^k \frac{dx^i}{dt} \frac{\partial f}{\partial x^i}, \quad (2.4)$$

Note the appearance of the geometrical tangent vector in this expression.

If we now define the differential operator

$$X = \sum_i v^i \frac{\partial}{\partial x^i} \quad \text{with} \quad v^i = \frac{dx^i(p(t))}{dt}, \quad \text{which is a tangent vector,} \quad (2.5)$$

then from Eqns.(2.5), (2.4) and (2.3), we can write

$$\begin{aligned} \frac{d}{dt}f(p(t)) &= \sum_{i=1}^k \frac{dx^i}{dt} \frac{\partial f}{\partial x^i}, \\ &\equiv Xf \end{aligned} \quad (2.6)$$

When $f = x^k(p(t))$ then

$$\frac{d}{dt}x^k(p(t)) = Xx^k(p(t)) \quad (2.7)$$

Thus the operator X acting on coordinate functions at a point gives the tangent vector at that point.

It is thus reasonable to think of the operator X itself as the tangent vector to the curve $p(t)$. A general tangent vector X can be constructed explicitly in terms of $\frac{\partial}{\partial x^i}$, ($i = 1, \dots, k$) with coefficients that need to be specified to describe a given vector field. The set of vector fields $\frac{\partial}{\partial x^i}$, ($i = 1, \dots, k$) thus form a basis for a vector field. Vector fields thus form a vector space of dimension equal to the dimension of the Euclidean space into which the local neighbourhood of the manifold is mapped. The basis vectors for the space are coordinate dependent.

General vector fields X and Y have important algebraic properties, which can be established using local representations. It is straightforward to check that the commutator $[X, Y] = XY - YX$ of two vector fields is also a vector field and that the commutator satisfies an identity known as the **Jacobi identity**.

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0 \quad (2.8)$$

§ **Problem:** Verify Jacobi identity for arbitrary vector fields X, Y and Z .

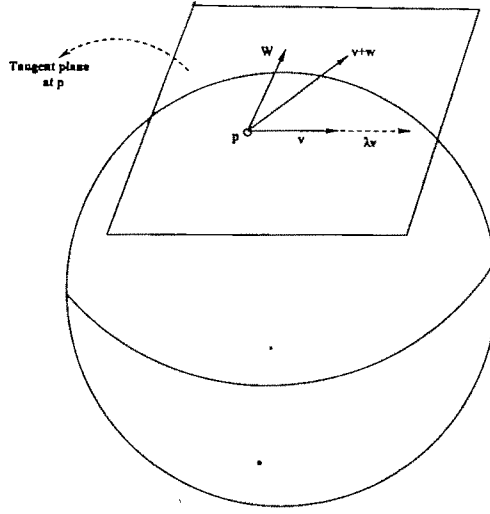
Tangent space

We can summarize our discussion on vector operators as follows. For every curve through a point $p \in \mathcal{M}$ there exists a tangent vector. So, one can define the tangent space $T_p(\mathcal{M})$ to the manifold $\mathcal{M}$ at the point p to be the space of all possible tangents at the point p . $T_p(\mathcal{M})$ describes a vector space and in local coordinates the set of differential operators

$$\left\{ \frac{\partial}{\partial x^i} \right\}, \quad i = 1, \dots, n, \quad (2.9)$$

forms a basis in $T_p(\mathcal{M})$.

Two arbitrary tangent vectors in terms of these bases can be added to give a new tangent vector and the scalar multiple of a tangent vector is also

Figure 2.1: Tangent plane at a point p to S^2

a tangent vector. Thus tangent vectors defined as operators form a vector space of dimension k . Explicitly

$$\begin{aligned} \sum_{i=1}^k (v^i \frac{\partial}{\partial x^i} + w^i \frac{\partial}{\partial x^i}) &= \sum_{i=1}^k (v^i + w^i) \frac{\partial}{\partial x^i} \\ \sum_{i=1}^k (\lambda v^i \frac{\partial}{\partial x^i}) &= \lambda \sum_{i=1}^k v^i \frac{\partial}{\partial x^i}. \end{aligned} \quad (2.10)$$

The figure (2.1) may be helpful.

Cotangent space

We now turn to the dual of tangent vectors - cotangent vectors. As we have suggested these will be differentials. We start by writing

$$df = \sum_{i=1}^k \frac{\partial f}{\partial x^i} dx^i, \quad (2.11)$$

we would like to make the functional dependence of df on a vector field X defined as the operator $X = \sum_i v^i \frac{\partial}{\partial x^i}$ precise by defining the pairing

$$\begin{aligned}
\langle df, X \rangle &= \left\langle \sum_i \frac{\partial f}{\partial x^i} dx^i, \sum_j v^j \frac{\partial}{\partial x^j} \right\rangle, \\
&= \sum_i \sum_j \frac{\partial}{\partial x^i} f v^j \left\langle dx^i, \frac{\partial}{\partial x^j} \right\rangle,
\end{aligned} \tag{2.12}$$

We now set

$$\begin{aligned}
\left\langle dx^i, \frac{\partial}{\partial x^j} \right\rangle &= \delta_j^i, \text{ i.e.} \\
&= 1, \text{ for } i = j, \\
&= 0, \text{ for } i \neq j.
\end{aligned} \tag{2.13}$$

Hence Eqn. (2.12) becomes

$$\begin{aligned}
\langle df, X \rangle &= \sum_i \sum_j \frac{\partial}{\partial x^i} f v^j \delta_j^i \\
&= \sum_i v^i \frac{\partial}{\partial x^i} f, \\
&= Xf \in \mathbb{R}
\end{aligned} \tag{2.14}$$

Eqn. (2.13) is a pairing that establishes dx^i as dual basis elements to the basis elements $\frac{\partial}{\partial x^i}$, while Eqn. (2.14) establishes the functional dependence of df on X .

We will use the following notation to write Eqn. (2.14):

$$\langle df, X \rangle \equiv df(X). \tag{2.15}$$

$$= Xf \tag{2.16}$$

With this idea of a differential established, it is possible to extend it by using the tools of multilinear algebra to define differential forms which in terms of local coordinates represent anti-symmetric tensor fields.

2.2.2 Differential Forms

Differential forms are anti-symmetric objects defined in a coordinate independent way on a manifold. Locally this is achieved by letting $\omega^{(p)}$ be an element of $\Lambda^{(p)}(V)$ with the basis vectors $\{dx^i\}$ as elements of the cotangent space. Thus elements of $\Lambda^{(p)}(V)$ can be written locally as

$$\omega^{(p)} = \sum \omega_{i_1, i_2, \dots, i_p}(x) dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_p}, \tag{2.17}$$

Note that in the summation each i_k is from $1, \dots, n$. This leads to many identical terms in the expression above, since any permutation of the labels $i_1, \dots, i_p$ give the same contribution. In view of the fact that an odd permutation will introduce a negative term in $\omega^{(p)}$, which will be compensated by a negative term from $dx^{i_1} \wedge \dots \wedge dx^{i_p}$. Thus, there are $p!$ equal contributions to $\omega^{(p)}$. Very often this feature is recognized by introducing a $\frac{1}{p!}$ term. Thus we can write

$$\begin{aligned}\omega^{(p)} &= \sum_{i_1 < \dots < i_p} \omega_{i_1, \dots, i_p}(x) dx^{i_1} \wedge \dots \wedge dx^{i_p}, \\ &= \frac{1}{p!} \sum_{i_1, \dots, i_p=1}^n \omega_{i_1, \dots, i_p}(x) dx^{i_1} \wedge \dots \wedge dx^{i_p}\end{aligned}\quad (2.18)$$

p -forms can be added and multiplied by scalars to generate new p -forms. An immediate consequence of the fact that they are elements of $\Lambda^{(p)}$. Since p and q -forms are elements of an exterior algebra, they can be multiplied (wedge product) to generate a $(p+q)$ -form. Explicitly

$$\begin{aligned}\omega^{(p)} \wedge \mu^{(q)} &\rightarrow \Lambda^{(p+q)}, \\ \omega^{(p)} \wedge \mu^{(q)} &= (-1)^{pq} \mu^{(q)} \wedge \omega^{(p)} : \text{non-commutative} \\ \mu_1^{q_1} \wedge (\mu_2^{q_2} \wedge \mu_3^{q_3}) &= (\mu_1^{q_1} \wedge \mu_2^{q_2}) \wedge \mu_3^{q_3} : \text{associative.}\end{aligned}\quad (2.19)$$

Remark: although the wedge product is not commutative, it does satisfy associativity.

The coefficients $\omega_{i_1, i_2, \dots, i_p}(x^i(p))$ in the local description of a p -form can be given a coordinate independent interpretation. They are simply the value of $\omega^{(p)}$ on p vectors. To see this, we consider the simple case of a one form $\omega^1 = \sum_{i=1}^n \omega_i(x(p)) dx^i$ and evaluate it on a vector X given by $X = \sum_{j=1}^n v^j \frac{\partial}{\partial x^j}$. Taking the pairing between vectors and differentials we can write

$$\begin{aligned}\omega^1(X) &\equiv \langle \omega^1, X \rangle, \\ &= \left\langle \sum_{i=1}^n \omega_i dx^i, \sum_{j=1}^n v^j \frac{\partial}{\partial x^j} \right\rangle, \\ &= \sum_{i=1}^n \omega_i v^i.\end{aligned}\quad (2.20)$$

Choosing $v^i = 1$, we see that ω_i is equal to $\omega^1(\frac{\partial}{\partial x^i})$. ω_i is in general a function of $x^i(p)$. For a general $\omega^{(p)}$

$$\omega_{i_1, \dots, i_p} = \omega^{(p)} \left(\frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_p}} \right) = \left\langle \omega^{(p)}, \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_p}} \right\rangle \quad (2.21)$$

♠ **Examples:** In two dimensions we have the following forms:

- Zero form: $f_0(x)$: function.
- One form: $f_1(x)dx^1 + f_2(x)dx^2$, and
- Two form: $f_3(x)dx^1 \wedge dx^2$.

In three dimensions we can have

- Zero form: f_0 ,
- One form: $f_1dx^1 + f_2dx^2 + f_3dx^3$,
- Two form: $g_1dx^1 \wedge dx^2 + g_2dx^3 \wedge dx^1 + g_3dx^2 \wedge dx^3$,
- Three form: $h dx^1 \wedge dx^2 \wedge dx^3$.

Now, let us return to the discussion of d operator. In two dimensions we can get

$$\begin{aligned}
 d(f_1dx^1 + f_2dx^2) &= \frac{\partial f_1}{\partial x^2}dx^2 \wedge dx^1 + \frac{\partial f_2}{\partial x^1}dx^1 \wedge dx^2, \\
 &= -\left(\frac{\partial f_1}{\partial x^2} - \frac{\partial f_2}{\partial x^1}\right)dx^1 \wedge dx^2, \\
 &= \left(\frac{\partial f_2}{\partial x^1} - \frac{\partial f_1}{\partial x^2}\right)dx^1 \wedge dx^2. \tag{2.22}
 \end{aligned}$$

Which is basically the “curl” (area).

Next consider the two form in dim 3:

$$\begin{aligned}
 &d(g_1dx^1 \wedge dx^2 + g_2dx^3 \wedge dx^1 + g_3dx^2 \wedge dx^3) \\
 &= \frac{\partial g_1}{\partial x^3}dx^3 \wedge dx^1 \wedge dx^2 + \frac{\partial g_2}{\partial x^2}dx^2 \wedge dx^3 \wedge dx^1 + \frac{\partial g_3}{\partial x^1}dx^1 \wedge dx^2 \wedge dx^3, \\
 &= \left(\frac{\partial g_3}{\partial x^1} + \frac{\partial g_2}{\partial x^2} + \frac{\partial g_1}{\partial x^3}\right)dx^1 \wedge dx^2 \wedge dx^3 \Rightarrow \text{“Divergence” operator.}
 \end{aligned}$$

§ **Problem:** Calculate $dx \wedge dy \wedge dz$ in polar and cylindrical coordinates.

2.2.3 Metric tensor

Besides anti-symmetric fields there is a specific symmetric tensor field $g_{ij}(x)$ of great importance in applications as well as in geometry. This is the metric tensor. The metric tensor $g_{ij}(x)$ is related to the square of the line element on a manifold as follows

$$g = ds^2 = \sum_{i,j} g_{ij}(x) dx^i dx^j. \quad (2.23)$$

A manifold on which it is possible to introduce g is called a Riemannian manifold and $g_{ij}(x)$ is the Riemannian metric of the manifold. A general theorem states that a Riemannian metric can be given to any paracompact manifold. Abstractly, the Riemannian line element built out of two basis elements dx^i, dx^j of the cotangent space is a tensor of type $(0, 2)$. As in the case of p -forms, the coordinate independent description for g_{ij} can be given by evaluating g on two vectors. Evaluation means

$$\begin{aligned} g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) &= \sum_{i,j} g_{ij} \left\langle dx^i dx^j, \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} \right\rangle \\ &= g_{ij}. \end{aligned} \quad (2.24)$$

As in the case of p -forms the metric tensor g is built out of elements of the cotangent space. It is assumed to be globally defined on the manifold. The expression given for g represents it in a local coordinate chart.

2.3 Operations on forms

2.3.1 Lie Derivative

The Lie derivative is related to transformations generated by a given vector field on a manifold. These transformations form an Abelian group. To see how this happens, consider a manifold which has translation invariance along the local coordinate direction x . This means that function $f(x)$ and $f(x+th)$ are equal. We note that $f(x+th) = e^{th \frac{d}{dx}} f(x)$. For an infinitesimal t , we have $f(x+th) = f(x) + th \frac{d}{dx} f(x)$. We recognize $h \frac{d}{dx}$ as a vector field X , which generates the symmetry. Furthermore it is also clear that e^{tX} is an element of an Abelian group. The group elements being labelled by t with multiplication rule $e^{tX} \cdot e^{sX} = e^{t+sX}$. The identity element corresponds to $t = 0$, while the inverse element of e^{tX} is e^{-tX} .

From the formula

$$\lim_{t \rightarrow 0} \frac{1}{t} (e^{tX} - 1) f(x) = Xf, \quad (2.25)$$

with

$$X = \sum_i f^i \frac{\partial}{\partial x^i}, \quad (2.26)$$

it follows that

$$\frac{dx^i}{dt} = f^i(x^1, \dots, x^n), \quad i = 1, \dots, n. \quad (2.27)$$

Note that the transformation group e^{tX} thus generates a curve on the manifold with tangent vectors prescribed by the vector field coefficients $f^i(x)$

Fixing the tangent vector thus leads to a set of n -ordinary differential equations. The existence theorem for differential equations guarantees a local solution for $x^i(t)$ under suitable conditions on f^i . Thus curves on the manifold with prescribed tangent vector exist.

2.3.2 Properties of the Lie derivative

We now proceed to determine the action of the Lie derivative on functions, vectors and forms on a manifold. This is done in a coordinate free manner. However, once such a result is established it can be written in terms of convenient coordinate system. In our discussion we will constantly move between these two approaches, i.e. a coordinate dependent and coordinate independent formulation.

Action of Lie derivative on a function

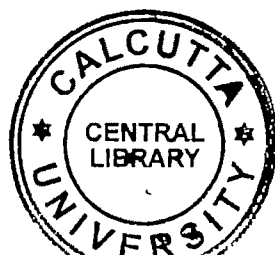
$$\mathcal{L}_X f = \lim_{t \rightarrow 0} \frac{1}{t} (e^{tX} - 1) f(x(p)) \quad (2.28)$$

where,

$$X = \sum_{i=1}^n v^i(x) \frac{\partial}{\partial x^i}. \quad (2.29)$$

Thus,

$$\mathcal{L}_X f = Xf, \quad (2.30)$$



Action of Lie derivative on a vector field

We next determine the action of $\mathcal{L}_X$ on a vector field Y . We suppose,

$$\begin{aligned}\mathcal{L}_X(Yf) &= (\mathcal{L}_X Y)f + Y(\mathcal{L}_X f), \\ \text{or, } (\mathcal{L}_X Y)f &= \mathcal{L}_X(Yf) - Y(\mathcal{L}_X f),\end{aligned}$$

where Yf is a function, so $\mathcal{L}_X(Yf) = XYf$, similarly $Y(\mathcal{L}_X f) = YXf$. Thus,

$$\begin{aligned}(\mathcal{L}_X Y)f &= XYf - YXf, \\ &= [X, Y]f,\end{aligned}$$

or,

$$\mathcal{L}_X Y = [X, Y]. \quad (2.31)$$

In doing this calculation we have assumed that the Lie derivative on the tensor product has the property

$$\mathcal{L}_X(T \otimes S) = (\mathcal{L}_X T) \otimes S + T \otimes (\mathcal{L}_X S). \quad (2.32)$$

We will establish the result shortly.

Action of Lie derivative on the metric tensor

We determine $\mathcal{L}_Z g(X, Y)$. We remember that $g(X, Y)$ is a function. It is basically the metric component g_{ij} .

$$\mathcal{L}_Z(g(X, Y)) = \mathcal{L}_Z(g, X \otimes Y). \quad (2.33)$$

The LHS of Eqn. (2.33) is

$$\mathcal{L}_Z(g(X, Y)) = Zg(X, Y), \quad (2.34)$$

while the RHS is

$$\mathcal{L}_Z(g, X \otimes Y) = (\mathcal{L}_Z g, X \otimes Y) + (g, \mathcal{L}_Z X \otimes Y) + (g, X \otimes \mathcal{L}_Z Y), \quad (2.35)$$

Now, from Eqn. (2.34, 2.35) and using $\mathcal{L}_Z Y = [Z, Y]$, we get,

$$\begin{aligned}Zg(X, Y) &= (\mathcal{L}_Z g, X \otimes Y) + (g, [Z, X] \otimes Y) + (g, X \otimes [Z, Y]), \\ \text{or, } (\mathcal{L}_Z g, X \otimes Y) &= Zg(X, Y) - (g, [Z, X] \otimes Y) - (g, X \otimes [Z, Y]).\end{aligned} \quad (2.36)$$

Now, let

$$X = \frac{\partial}{\partial x^i}, \quad Y = \frac{\partial}{\partial x^j}, \quad Z = \sum_{a=1}^n v^a(x) \frac{\partial}{\partial x^a}. \quad (2.37)$$

If $\mathcal{L}_Z$ is the isometry (i.e. it does not change the metric), then

$$(\mathcal{L}_Z g)(X, Y) = 0 \quad (2.38)$$

then Eqn. (2.36) is called the Killing Equation. This is an important result which determines the symmetries of a given metric g_{ij} . Explicitly

$$\begin{aligned} 0 &= Zg(X, Y) - (g, [Z, X] \otimes Y) - (g, X \otimes [Z, Y]), \\ &= \sum_{a=1}^n v^a(x) \frac{\partial g_{ij}(x)}{\partial x^a} - g([Z, X], Y) - g(X, [Z, Y]) \end{aligned} \quad (2.39)$$

We proceed to write out this equation in terms of coordinates. We have to evaluate 2nd and 3rd term of the right hand side of (2.39). For that, we will use the following lemma:

$$[AB, C] = [A, C]B + A[B, C]. \quad (2.40)$$

Hence using (2.40)

$$\begin{aligned} [Z, X] &= \sum_{a=1}^n \left[v^a(x) \frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^i} \right], \\ &= \sum_{a=1}^n \left\{ \left[v^a(x), \frac{\partial}{\partial x^i} \right] \frac{\partial}{\partial x^a} + v^a(x) \underbrace{\left[\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^i} \right]}_{=0} \right\}. \end{aligned} \quad (2.41)$$

The first term of (2.41) can be written as

$$\begin{aligned} \left[v^a(x), \frac{\partial}{\partial x^i} \right] f &= \left(v^a(x) \frac{\partial}{\partial x^i} - \frac{\partial}{\partial x^i} v^a(x) \right) f, \\ &= v^a(x) \frac{\partial f}{\partial x^i} - \frac{\partial}{\partial x^i} (v^a(x) f), \\ &= - \left(\frac{\partial v^a(x)}{\partial x^i} \right) f \end{aligned} \quad (2.42)$$

So,

$$\left[v^a(x), \frac{\partial}{\partial x^i} \right] = - \left(\frac{\partial v^a(x)}{\partial x^i} \right). \quad (2.43)$$

Hence from Eqn. (2.41, 2.43),

$$[Z, X] = - \sum_{a=1}^n \left(\frac{\partial v^a(x)}{\partial x^i} \right) \frac{\partial}{\partial x^a}. \quad (2.44)$$

$$\text{and, } [Z, Y] = - \sum_{a=1}^n \left(\frac{\partial v^a(x)}{\partial x^j} \right) \frac{\partial}{\partial x^a} \quad (2.45)$$

So, finally we get

$$\begin{aligned} 0 &= \sum_{a=1}^n v^a(x) \frac{\partial g_{ij}(x)}{\partial x^a} + \sum_{a=1}^n g \left(\frac{\partial v^a(x)}{\partial x^i} \frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^j} \right) \\ &\quad + \sum_{a=1}^n g \left(\frac{\partial}{\partial x^i}, \frac{\partial v^a(x)}{\partial x^j} \frac{\partial}{\partial x^a} \right) \\ \text{or, } &= \sum_{a=1}^n \left[v^a(x) \frac{\partial g_{ij}(x)}{\partial x^a} + \frac{\partial v^a}{\partial x^i} g_{aj}(x) + \frac{\partial v^a}{\partial x^j} g_{ia}(x) \right] \end{aligned} \quad (2.46)$$

♠ **Example:** As a simple example of this equation, we suppose that $g_{ij} = \delta_{ij}$. This gives

$$\frac{\partial v^j}{\partial x^i} + \frac{\partial v^i}{\partial x^j} = 0 \quad (2.47)$$

and leads to vector fields that keep g invariant. These are translations and rotations.

Lie derivative on one forms

We proceed to determine the action of the Lie derivative on a p form.

$$\begin{aligned} \mathcal{L}_Z(\omega, X) &= (\mathcal{L}_Z\omega, X) + (\omega, \mathcal{L}_ZX), \\ \text{now, } (\mathcal{L}_Z\omega, X) &= \mathcal{L}_Z(\omega X) - (\omega, \mathcal{L}_ZX), \end{aligned} \quad (2.48)$$

We recall that locally for a one form (we use the notations of Eqn (46) for X and Z),

$$\begin{aligned} (\omega, X) &= \sum_{a=1}^n \left(\omega_a dx^a, \frac{\partial}{\partial x^i} \right) = \omega_i(x), \\ \text{hence, } \mathcal{L}_Z(\omega X) &= Z\omega_i(x) \end{aligned} \quad (2.49)$$

So, Eqn (2.48) becomes

$$(\mathcal{L}_Z \omega, X) = Z\omega_i(x) - (\omega, [Z, X]), \quad (2.50)$$

$$\begin{aligned} &= Z\omega_i(x) + \sum_{a=1}^n \left(\omega_j dx^j, \frac{\partial v^a}{\partial x^i} \frac{\partial}{\partial x^a} \right), \quad [\text{Using Eqn (2.44)}] \\ &= \sum_{a=1}^n \left(v^a \frac{\partial \omega_i(x)}{\partial x^a} + \omega_a(x) \frac{\partial v^a}{\partial x^i} \right) \end{aligned} \quad (2.51)$$

For a general p -form the corresponding result is

$$\begin{aligned} (\mathcal{L}_Z \omega^{(p)})(X_1, \dots, X_p) &= Z(\omega^{(p)}(X_1, \dots, X_p)) \\ &\quad - \sum_{k=1}^p \omega^{(p)}(X_1, \dots, [Z, X_k], \dots, X_p). \end{aligned} \quad (2.52)$$

A simple consequence of this result is

$$\mathcal{L}_X(T \otimes S) = (\mathcal{L}_X T) \otimes S + T \otimes (\mathcal{L}_X S). \quad (2.53)$$

We note that

$$\begin{aligned} &(\mathcal{L}_Z(T^{(p)} \otimes S^{(q)}))(X_1, \dots, X_{p+q}) \\ &= Z((T^{(p)} \otimes S^{(q)})(X_1, \dots, X_{p+q})) - \\ &\quad \sum_{k=1}^{p+q} (T^{(p)} \otimes S^{(q)})(X_1, \dots, [Z, X_k], \dots, X_{p+q}). \end{aligned}$$

from the result just stated. On the other hand by definition

$$(T^{(p)} \otimes S^{(q)})(X_1, \dots, X_{p+q}) = T^{(p)}(X_1, \dots, X_p) S^{(q)}(X_{p+1}, \dots, X_{p+q}).$$

Hence

$$\begin{aligned} &Z((T^{(p)} \otimes S^{(q)})(X_1, \dots, X_{p+q})) \\ &= Z(T^{(p)}(X_1, \dots, X_p) S^{(q)}(X_{p+1}, \dots, X_{p+q})) + \\ &\quad T^{(p)}(X_1, \dots, X_p) Z(S^{(q)}(X_{p+1}, \dots, X_{p+q})). \end{aligned}$$

while the term with the commutator splits into two terms depending on whether $k < p$ or $k > p$, i.e.

$$\begin{aligned} &= - \sum_{k=1}^p T^{(p)}(X_1, \dots, [Z, X_k], \dots, X_p) S^{(q)}(X_{p+1}, \dots, X_{p+q}) \\ &\quad - T^{(p)}(X_1, \dots, X_p) \left(\sum_{k=p+1}^{p+q} S^{(q)}(X_{p+1}, \dots, [Z, X_k], \dots, X_{p+q}) \right), \end{aligned}$$

which establishes the result.

A consequence of this result is

$$\mathcal{L}_Z(\omega^{(p)} \wedge v^{(q)}) = \mathcal{L}_Z\omega^{(p)} \wedge v^{(q)} + \omega^{(p)} \wedge \mathcal{L}_Z v^{(q)}. \quad (2.54)$$

This equation says that the Lie derivative acts on the wedge product of forms just as the derivative acts on the product of functions.

§ **Problem:** Prove that $\mathcal{L}_{[X,Y]} = [\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_X\mathcal{L}_Y - \mathcal{L}_Y\mathcal{L}_X$.

§ **Problem:** Prove that $d\mathcal{L}_X = \mathcal{L}_Xd$.

2.4 Three new operations: d , i_X and $\star$:

We now introduce operators which modify the order of a form. The three new operators we consider are

- The exterior derivative (d) : $d\omega^{(p)} \rightarrow \omega^{(p+1)}$
- the interior product (i_X) : $i_X\omega^{(p)} \rightarrow \omega^{(p-1)}$ and
- the Hodge star ($\star$) : $\star\omega^{(p)} \rightarrow \omega^{(n-p)}$, where $\dim \mathcal{M} = n$.

An important property of a differential form is that it is metric independent. The operators d and i_X which relate forms of different order are also metric independent. However the Hodge star operator which relates a p -form to its “dual”: an $(n-p)$ -form where n is the dimension of the manifold is metric dependent. The metric independent nature of forms has been used to study topological features of a manifold. We will briefly examine this feature as a comment following a proof of the Poincaré lemma.

2.4.1 The exterior derivative (d):

The exterior derivative maps a p -form to a $(p+1)$ -form. Locally a p -form is represented in terms of the wedge product of p -terms. The action of the d -operator increases the number of such terms by one in the following way

$$d\omega^{(p)} = \sum_{i_1, \dots, i_p, j} \frac{\partial \omega_{i_1, \dots, i_p}^{(p)}(x)}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_p}. \quad (2.55)$$

An immediate important consequence of this representation is the result

$$d^2 = 0. \quad (2.56)$$

To see this, we simply use the definition of the d -operator and note that

$$d(d\omega^{(p)}) = \sum_{i_1, \dots, i_p, j_1, j_2} \underbrace{\frac{\partial^2 \omega_{i_1, \dots, i_p}(x)}{\partial x^{j_1} \partial x^{j_2}}}_{\text{symmetric in } j_1, j_2} \underbrace{dx^{j_1} \wedge dx^{j_2} dx^{i_1} \wedge \dots \wedge dx^{i_p}}_{\text{antisymmetric } (p+2) \text{ form}}. \quad (2.57)$$

The right hand side is zero as the second derivative term is symmetric under the exchange of j_1, j_2 while the wedge product factor is anti-symmetric.

$$\boxed{d(d\omega^{(p)}) = 0 \Rightarrow d^2 = 0} \quad (2.58)$$

This result suggests the following question: if $d\omega^{(p)} = 0$, is it always the case that $\omega^{(p)} = d\eta^{(p-1)}$ for some $(p-1)$ -form η . We will see that the answer is yes locally, but not globally. The local result is known as the Poincaré lemma while the possibility of the result not holding due to global considerations is a subject matter of de Rham cohomology. We proceed to look at these two aspects. We want to think of a tangent vector field as differential operator with coefficients that are functions of the coordinates. with Poincaré lemma.

2.4.2 Poincaré Lemma

The essential idea of the proof is that a given local description for a p -form can be deformed to zero by introducing a scale (homotopy) parameter t . This can only be done if the space is locally contractible.

We can write $\omega^{(p)}$ locally as

$$\omega^{(p)} = \omega_{i_1, \dots, i_p}(x^1, \dots, x^p) dx^{i_1} \wedge \dots \wedge dx^{i_p} \quad (2.59)$$

Now, we introduce the homotopy parameter t and define

$$\begin{aligned} \widehat{\omega}^{(p)} &= \sum \omega_{i_1, \dots, i_p}(tx^1, \dots, tx^p) dx^{i_1} \wedge \dots \wedge dx^{i_p} t^p \\ &+ \sum \gamma_{i_1, \dots, \widehat{i_k}, \dots, i_p}(x^1, \dots, x^p) dt \wedge dx^{i_1} \wedge \dots \\ &\wedge \widehat{dx^{i_k}} \wedge \dots \wedge dx^{i_p}, \end{aligned} \quad (2.60)$$

where $0 \leq t \leq 1$. Note that $\widehat{\omega}^{(p)}|_{(t=0)} = 0$ and $\widehat{\omega}^{(p)}|_{(t=1)} = \omega^{(p)}$, which follows from the fact that at the end points $t = 0$ and $t = 1$, $dt = 0$. As usual the notation $\widehat{dx^{i_k}}$ means that this particular element is missing from the expression. Note that $d\omega^{(p)} = 0$ implies $d\widehat{\omega}^{(p)} = 0$ which follows from the property of d -operator under the maps, since $\phi(d\omega) = d(\phi\omega)$. Hence we can

write

$$\begin{aligned}
 0 = d\widehat{\omega}^{(p)} &= \frac{\partial}{\partial t}(\omega_{i_1, \dots, i_p}(tx^1, \dots, tx^p)t^p)dt \wedge dx^{i_1} \wedge \dots \wedge dx^{i_p} \\
 &\quad + \frac{\partial}{\partial x^{i_k}}(\gamma_{i_1, \dots, i_p})dx^{i_k} \wedge dt \wedge dx^{i_1} \wedge \dots \wedge dx^{i_p} \\
 0 &= \frac{\partial}{\partial t}(\omega_{i_1, \dots, i_p}(tx^1, \dots, tx^p)t^p)dt \wedge dx^{i_1} \wedge \dots \wedge dx^{i_p} \\
 &\quad + (-1)^k \frac{\partial}{\partial x^{i_k}}(\gamma_{i_1, \dots, i_p})dt \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k} \wedge \dots \wedge dx^{i_p} \\
 0 &= \frac{\partial}{\partial t}(\omega_{i_1, \dots, i_p}(tx^1, \dots, tx^p)t^p)dt \wedge dx^{i_1} \wedge \dots \wedge dx^{i_p} \\
 &\quad - (-1)^{k-1} \frac{\partial}{\partial x^{i_k}}(\gamma_{i_1, \dots, i_p})dt \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k} \wedge \dots \wedge dx^{i_p}
 \end{aligned}$$

Hence we can write finally

$$\int_0^1 dt \frac{\partial}{\partial t}(\widehat{\omega}_{i_1, \dots, i_p}(tx^1, \dots, tx^p)t^p) = \int_0^1 (-1)^{k-1} \frac{\partial}{\partial x^{i_k}}(\gamma_{i_1, \dots, i_p})dt \quad (2.61)$$

We want to think of a tangent vector field as differential operator with coefficients that are functions of the coordinates. In the previous step we have used the fact that $\widehat{\omega}^{(p)}(t=1) = \omega^{(p)}$ and, $\widehat{\omega}^{(p)}(t=0) = 0$. So, explicitly

$$\omega_{i_1, \dots, i_p}(x^1, \dots, x^p) = \int_0^1 (-1)^{k-1} \frac{\partial}{\partial x^{i_k}}(\gamma_{i_1, \dots, \widehat{i_k}, \dots, i_p})dt \quad (2.62)$$

So, we conclude that

$$\begin{aligned}
 \omega^{(p)} &= d \left\{ \left(\int_0^1 dt \gamma_{i_1, \dots, \widehat{i_k}, \dots, i_p} \right) dx^{i_1} \dots \widehat{dx^{i_k}} \dots dx^{i_p} \right\}, \\
 &= d\eta^{(p-1)}
 \end{aligned} \quad (2.63)$$

2.4.3 A brief discussion on de-Rham cohomology

We now proceed to show that Poincaré's lemma need not hold globally. To do this it is natural to introduce two spaces, namely $\mathcal{Z}^p$ and $\mathcal{B}^p$ which consists of all closed and exact p -forms respectively on $\mathcal{M}$ defined as follows

$$\begin{aligned}
 \mathcal{Z}^p &: \{ \omega^{(p)} \mid d\omega^{(p)} = 0 \} \Rightarrow \text{this is called space of all closed } p\text{-forms,} \\
 \mathcal{B}^p &: \{ \omega^{(p)} \mid \omega^{(p)} = d\eta^{(p-1)} \} \Rightarrow \text{this is called space of all exact } p\text{-form.}
 \end{aligned}$$

Since exactness includes closeness, we can write

$$\mathcal{B}^p \subset \mathcal{Z}^p \quad (2.64)$$

One can define the quotient space as

$$\begin{aligned}\mathcal{H}^{(p)}(\mathcal{M}, \mathbb{R}) &= \mathcal{Z}^p(\mathcal{M}, \mathbb{R}) / \mathcal{B}^p(\mathcal{M}, \mathbb{R}) \\ &= p\text{-th de Rham cohomology group}\end{aligned}$$

The group property can be seen easily if we suppose that the number of closed p -forms on a manifold is finite. One can then choose a basis of these forms. The coefficients of these basis p -forms can be chosen as arbitrary real numbers, i.e. they are elements of $\mathbb{R}$. Elements of $\mathbb{R}$ form an Abelian group under addition with zero as its identity element. Thus one can associate with $\mathcal{Z}^p$ and $\mathcal{B}^p$, Abelian groups which reflect the number of elements that are closed or exact. The quotient of these groups is the de Rham cohomology group. If Poincaré's lemma was globally valid, the spaces $\mathcal{Z}^p$ and $\mathcal{B}^p$ would be equal and the cohomology group would be trivial, i.e. it consists of a single element corresponding to the identity element of $\mathbb{R}$, which is zero. We now give an example where this is not the case. This is the case of the one dimensional manifold S^1 . For this manifold zero forms and one forms are only possible. An arbitrary zero form will be described by a function parametrized by an angle θ which is periodic, while a one form can be written as $\omega^{(1)} = f_1(\theta)d\theta$, again with $f_1(\theta)$ being periodic in θ . We now examine the space $\mathcal{Z}^1$ and $\mathcal{B}^1$.

$$\begin{aligned}\mathcal{Z}^1 &= \{\omega^{(1)} \mid d\omega^{(1)} = 0\}, \\ \mathcal{B}^1 &= \{\omega^{(1)} \mid \omega^{(1)} = d\omega^{(0)}\}.\end{aligned}\tag{2.65}$$

Let, $\omega^{(1)} = f_1(\theta)d\theta$, where $0 \leq \theta \leq 2\pi$. We write $f_1(\theta)$ as

$$f_1(\theta) = \sum_{n=-\infty}^{\infty} c_1^n e^{in\theta}.\tag{2.66}$$

Then $d\omega^{(1)} = 0$ trivially. We note from the above equations that

$$\begin{aligned}\omega^{(0)} &= \sum_{n=-\infty}^{\infty} c_0^n e^{in\theta}, \\ \text{or, } d\omega^{(0)} &= \sum_{n=-\infty}^{\infty} (in)c_0^n e^{in\theta} d\theta.\end{aligned}\tag{2.67}$$

Since we want to find out all the exact forms, we need $\omega^{(1)} = d\omega^{(0)}$. Hence comparing coefficients, we get

$$c_1^n = c_0^n(in), \text{ for } n \neq 0,\tag{2.68}$$

which means that c_0^n , which is a real number can not be eliminated by a term coming from $d\omega^{(0)}$. Thus we can write

$$\begin{aligned}\mathcal{H}^{(1)}(S^1, \mathbb{R}) &= \mathcal{Z}^1(S^1, \mathbb{R})/\mathcal{B}^1(S^1, \mathbb{R}) \\ &= c_n^0.\end{aligned}$$

Thus we write

$$\mathcal{H}^1(S^1, \mathbb{R}) = \mathbb{R}. \quad (2.69)$$

Here $\mathcal{Z}^1 \neq \mathcal{B}^1$, i.e. Poincaré lemma doesnot extend globally in this case.

2.4.4 Coordinate free expression for d

Following our general procedure of establishing results locally and then recasting these results in a coordinate free way, we determine the action of the d -operator on a p -form in a coordinate free way.

If ω is a p -form on $\mathcal{M}$, then there is a unique $(p+1)$ -form $d\omega$ on $\mathcal{M}$ such that for every set of vector fields $X_1, \dots, X_{p+1}$ we have

$$\begin{aligned}d\omega^{(p)}(X_1, X_2, \dots, X_{p+1}) &= \sum_{i=1}^{p+1} (-1)^{i+1} X_i \omega(X_1, \dots, \widehat{X_i}, \dots, X_{p+1}) \\ &+ \sum_{1 \leq i < j \leq p+1} (-1)^{i+j} \omega([X_i, X_j], X_1, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_{p+1}),\end{aligned}$$

where a "caret" $\widehat{}$ over X_i indicates that it is omitted.

For the case of 1-form, i.e. $p = 1$, we have

$$d\omega^{(1)}(X, Y) = X\omega^{(1)}(Y) - Y\omega^{(1)}(X) - \omega^{(1)}([X, Y]). \quad (2.70)$$

This result will be used when we discuss Frobenius' theorem. Let us prove this result.

In the local representation, let

$$X = \sum_{a=1}^n f^a(x) \frac{\partial}{\partial x^a}, \quad Y = \sum_{b=1}^n g^b(x) \frac{\partial}{\partial x^b}, \quad \omega^{(1)} = \sum_{i=1}^n \omega_i(x) dx^i. \quad (2.71)$$

Hence we can write

$$\begin{aligned}
 \omega^{(1)}(X) &= \left\langle \sum_{i=1}^n \omega_i(x) dx^i, \sum_{a=1}^n f^a(x) \frac{\partial}{\partial x^a} \right\rangle, \\
 &= \sum_{i=1}^n \sum_{a=1}^n \omega_i(x) f^a(x) \underbrace{\left\langle dx^i, \frac{\partial}{\partial x^a} \right\rangle}_{=\delta_a^i}, \\
 &= \sum_{a=1}^n \omega_a(x) f^a(x).
 \end{aligned} \tag{2.72}$$

Similarly, one can show that

$$\omega^{(1)}(Y) = \sum_{b=1}^n \omega_b(x) g^b(x). \tag{2.73}$$

Next, we have to calculate $\omega^{(1)}([X, Y])$. For that, let us first note the following lemma regarding the commutator of operators:

$$\begin{aligned}
 [AB, C] &= ABC - CAB, \\
 &= ABC - ACB + ACB - CAB, \\
 &= A(BC - CB) + (AC - CA)B, \\
 &= A[B, C] + [A, C]B,
 \end{aligned} \tag{2.74}$$

$$\text{Similarly, } [A, BC] = [A, B]C + B[A, C]. \tag{2.75}$$

Now, let us calculate $[X, Y]$ first:

$$\begin{aligned}
 [X, Y] &= \left[\sum_{a=1}^n f^a(x) \frac{\partial}{\partial x^a}, \sum_{b=1}^n g^b(x) \frac{\partial}{\partial x^b} \right], \\
 &= \sum_{a=1}^n \sum_{b=1}^n \left[f^a(x) \frac{\partial}{\partial x^a}, g^b(x) \frac{\partial}{\partial x^b} \right].
 \end{aligned} \tag{2.76}$$

We consider the term inside the bracket separately :

$$\begin{aligned}
 & \left[f^a(x) \frac{\partial}{\partial x^a}, g^b(x) \frac{\partial}{\partial x^b} \right] \\
 = & f^a(x) \left[\frac{\partial}{\partial x^a}, g^b(x) \frac{\partial}{\partial x^b} \right] + \left[f^a(x), g^b(x) \frac{\partial}{\partial x^b} \right] \frac{\partial}{\partial x^a}, \quad [\text{Using (2.74)}] \\
 = & f^a(x) \left[\frac{\partial}{\partial x^a}, g^b(x) \right] \frac{\partial}{\partial x^b} + f^a(x) g^b(x) \left[\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b} \right] \\
 & + g^b(x) \left[f^a(x), \frac{\partial}{\partial x^b} \right] \frac{\partial}{\partial x^a} + [f^a(x), g^b(x)] \frac{\partial}{\partial x^b} \frac{\partial}{\partial x^a} \quad (2.77)
 \end{aligned}$$

$$= f^a(x) \left[\frac{\partial}{\partial x^a}, g^b(x) \right] \frac{\partial}{\partial x^b} + g^b(x) \left[f^a(x), \frac{\partial}{\partial x^b} \right] \frac{\partial}{\partial x^a} \quad (2.78)$$

here, we have used Eqn. (2.75) in the third line. We also note that the second and the last term of the expression (2.77) are zero.

Now, we can separately evaluate the terms in (2.78). We note

$$\begin{aligned}
 \left[\frac{\partial}{\partial x^a}, g^b(x) \right] \psi &= \frac{\partial}{\partial x^a} (g^b(x) \psi) - g^b(x) \frac{\partial \psi}{\partial x^a} \\
 &= \left(\frac{\partial g^b(x)}{\partial x^a} \right) \psi. \quad (2.79)
 \end{aligned}$$

Hence

$$\left[\frac{\partial}{\partial x^a}, g^b(x) \right] = \frac{\partial g^b(x)}{\partial x^a}, \quad (2.80)$$

$$\text{similarly, } \left[f^a(x), \frac{\partial}{\partial x^b} \right] = -\frac{\partial f^a(x)}{\partial x^b}. \quad (2.81)$$

Using the expression (2.80) and (2.81) in (2.78):

$$\left[f^a(x) \frac{\partial}{\partial x^a}, g^b(x) \frac{\partial}{\partial x^b} \right] = \left(f^a(x) \frac{\partial g^b(x)}{\partial x^a} \right) \frac{\partial}{\partial x^b} - \left(g^b(x) \frac{\partial f^a(x)}{\partial x^b} \right) \frac{\partial}{\partial x^a} \quad (2.82)$$

Since the indices a and b are dummy we can interchange a with b in the first term of r.h.s of (2.82) and then taking the sum, Eqn. (2.76) becomes:

$$[X, Y] = \sum_{a=1}^n \sum_{b=1}^n \left(f^b(x) \frac{\partial g^a(x)}{\partial x^b} - g^b(x) \frac{\partial f^a(x)}{\partial x^b} \right) \frac{\partial}{\partial x^a} \quad (2.83)$$

Now,

$$\begin{aligned}
 \omega^{(1)}([X, Y]) &= \left\langle \sum_{i=1}^n \omega_i(x) dx^i, [X, Y] \right\rangle, \\
 &= \left\langle \sum_{i=1}^n \omega_i(x) dx^i, \sum_{a=1}^n \sum_{b=1}^n \left(f^b(x) \frac{\partial g^a(x)}{\partial x^b} - g^b(x) \frac{\partial f^a(x)}{\partial x^b} \right) \frac{\partial}{\partial x^a} \right\rangle, \\
 &= \sum_{i=1}^n \sum_{a=1}^n \sum_{b=1}^n \omega_i(x) \left(f^b(x) \frac{\partial g^a(x)}{\partial x^b} - g^b(x) \frac{\partial f^a(x)}{\partial x^b} \right) \underbrace{\left\langle dx^i, \frac{\partial}{\partial x^a} \right\rangle}_{\hookrightarrow \delta_a^i}, \\
 &= \sum_{a=1}^n \sum_{b=1}^n \omega_a(x) \left(f^b(x) \frac{\partial g^a(x)}{\partial x^b} - g^b(x) \frac{\partial f^a(x)}{\partial x^b} \right) \quad (2.84)
 \end{aligned}$$

Now we will find what $d\omega^{(1)}(X, Y)$ is.

$$\begin{aligned}
 d\omega^{(1)}(X, Y) &= \left\langle \sum_{i=1}^n \sum_{j=1}^n \frac{\partial \omega_i(x)}{\partial x^j} dx^j \wedge dx^i, \sum_{a=1}^n f^a(x) \frac{\partial}{\partial x^a} \otimes \sum_{b=1}^n g^b(x) \frac{\partial}{\partial x^b} \right\rangle \\
 &= \sum_{i=1}^n \sum_{j=1}^n \sum_{a=1}^n \sum_{b=1}^n \frac{\partial \omega_i(x)}{\partial x^j} f^a(x) g^b(x) \underbrace{\left\langle (dx^j \otimes dx^i - dx^i \otimes dx^j), \frac{\partial}{\partial x^a} \otimes \frac{\partial}{\partial x^b} \right\rangle}_{\hookrightarrow \delta_a^j \delta_b^i - \delta_a^i \delta_b^j}
 \end{aligned}$$

Hence

$$d\omega^{(1)}(X, Y) = \sum_{a=1}^n \sum_{b=1}^n \left[\frac{\partial \omega_b(x)}{\partial x^a} f^a(x) g^b(x) - \frac{\partial \omega_a(x)}{\partial x^b} f^a(x) g^b(x) \right] \quad (2.85)$$

This can also be written as

$$\begin{aligned}
 d\omega^{(1)}(X, Y) &= \sum_{a=1}^n \sum_{b=1}^n \left\{ f^a(x) \frac{\partial}{\partial x^a} \{ \omega_b(x) g^b(x) \} - g^b(x) \frac{\partial}{\partial x^b} \{ \omega_a(x) f^a(x) \} \right. \\
 &\quad \left. - f^a(x) \omega_b(x) \frac{\partial g^b(x)}{\partial x^a} + g^b(x) \omega_a(x) \frac{\partial f^a(x)}{\partial x^b} \right\} \\
 &= \sum_{a=1}^n \sum_{b=1}^n \left\{ f^a(x) \frac{\partial}{\partial x^a} \{ \omega_b(x) g^b(x) \} - g^b(x) \frac{\partial}{\partial x^b} \{ \omega_a(x) f^a(x) \} \right. \\
 &\quad \left. - \left(f^b(x) \frac{\partial g^a(x)}{\partial x^b} - g^b(x) \frac{\partial f^a(x)}{\partial x^b} \right) \omega_a(x) \right\} \quad (2.86)
 \end{aligned}$$

We can use the expressions (66), (67), (68) and (79) to write

$$\boxed{d\omega^{(1)}(X, Y) = X\omega^{(1)}(Y) - Y\omega^{(1)}(X) - \omega^{(1)}([X, Y])}$$

Action of d -operator on wedge products

By using the local expression for the action of d , we quickly establish the result

$$\boxed{d(\omega^{(p)} \wedge \mu^{(q)}) = (d\omega^{(p)}) \wedge \mu^{(q)} + (-1)^p \omega^{(p)} \wedge (d\mu^{(q)})} \quad (2.87)$$

To prove this we write

$$\begin{aligned} d(\omega^{(p)} \wedge \mu^{(q)}) &= d\left(\sum \omega_{i_1, \dots, i_p} \mu_{i_{p+1}, \dots, i_{p+q}} dx^{i_1} \wedge \dots \right. \\ &\quad \left. \wedge dx^{i_p} \wedge dx^{i_{p+1}} \wedge \dots \wedge dx^{i_{p+q}}\right) \\ &= \sum \frac{\partial}{\partial x^k} (\omega_{i_1, \dots, i_p} \mu_{i_{p+1}, \dots, i_{p+q}}) dx^k \wedge dx^{i_1} \wedge \dots \\ &\quad \wedge dx^{i_p} \wedge dx^{i_{p+1}} \wedge \dots \wedge dx^{i_{p+q}} \\ &= \sum \frac{\partial}{\partial x^k} (\omega_{i_1, \dots, i_p}) \mu_{i_{p+1}, \dots, i_{p+q}} dx^k \wedge dx^{i_1} \wedge \dots \\ &\quad \wedge dx^{i_p} \wedge dx^{i_{p+1}} \wedge \dots \wedge dx^{i_{p+q}} \\ &\quad + \sum \omega_{i_1, \dots, i_p} \frac{\partial}{\partial x^k} (\mu_{i_{p+1}, \dots, i_{p+q}}) dx^k \wedge dx^{i_1} \wedge \dots \\ &\quad \wedge dx^{i_p} \wedge dx^{i_{p+1}} \wedge \dots \wedge dx^{i_{p+q}} \\ &= (d\omega^{(p)}) \wedge \mu^{(q)} + (-1)^p \omega^{(p)} \wedge (d\mu^{(q)}) \end{aligned} \quad (2.88)$$

since the first factor is clearly $(d\omega^{(p)}) \wedge \mu^{(q)}$, while to write the second factor as $d\mu^{(q)}$ requires moving the dx^k factor across the set $dx^{i_1} \wedge \dots \wedge dx^{i_p}$. This introduces a factor $(-1)^p$.

Note the difference between the action of the d -operator on the wedge product of forms and that of the Lie derivative. For the Lie derivative there is no $(-1)^p$ factor present.

Behavior of the d -operator under maps

A remarkable property of the exterior derivative operator d is that for a smooth map from a manifold $\phi : \mathcal{M} \rightarrow \mathcal{N}$, where $\dim \mathcal{M} \geq \dim \mathcal{N}$, then

applying the map first to a form in $\mathcal{M}$ and then acting with d on the resultant form in $\mathcal{N}$ gives the same result as applying d first on a form and then mapping it to $\mathcal{N}$. The operations of mapping and exterior differentiation commute. An inductive proof of this result using a local representation for $\omega^{(p)}$ can be given. Let $\omega^{(p)} = \omega_{i_1, \dots, i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p}$.

$$\begin{aligned} d\phi\omega^{(p)} &= d\phi(\omega_{i_1, \dots, i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p}), \\ &= d\phi((\omega_{i_1, \dots, i_p} dx^{i_1} \wedge \dots \wedge dx^{i_{p-1}}) \wedge dx^{i_p}), \end{aligned}$$

we use induction on the order of a form, i.e. we assume the result is true for $(p-1)$ -form.

$$\begin{aligned} &= d(\phi(\omega_{i_1, \dots, i_p} dx^{i_1} \wedge \dots \wedge dx^{i_{p-1}}) \wedge \phi(dx^{i_p}), \\ &= \phi(d\omega_{i_1, \dots, i_p} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_{p-1}}) \wedge \phi(dx^{i_p}), \text{ (using } d^2 = 0) \\ &= \phi(d\omega_{i_1, \dots, i_p} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_{p-1}} \wedge dx^{i_p}), \\ &= \phi(d\omega^{(p)}). \end{aligned} \tag{2.89}$$

Note, we have assumed that the result is true for $p = 0$, this needs to be checked.

2.4.5 Differential forms and volume

A natural definition for the infinitesimal volume element is to require that it be proportional to $\epsilon_{1\dots n} e^1 \wedge \dots \wedge e^n$ if the differentials e^i represent orthogonal directions in the space, i.e. if the corresponding metric can be written as

$$ds^2 = (e^1)^2 + \dots + (e^n)^2. \tag{2.90}$$

The constant epsilon tensor introduced reflects the orientation of the volume element. We would like to determine the infinitesimal volume element defined for the general line element $ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$, which follows from this natural definition.

$$ds^2 = \sum_{\mu, \nu} g_{\mu\nu}(x) dx^\mu dx^\nu,$$

The trick is to write

$$ds^2 = \sum_a (e^a)^2,$$

by setting

$$e^a = \sum_{\mu} e_{\mu}^a dx^{\mu},$$

so that,

$$g_{\mu\nu} = \sum_a e_{a\mu} e_{a\nu} \quad (2.91)$$

where, $\mu, \nu = 1, \dots, n$ and $a, b = 1, \dots, n$. The volume element is given by

$$\begin{aligned} e^1 \wedge \dots \wedge e^n &= e_{\mu_1}^1 \dots e_{\mu_n}^n dx^{\mu_1} \wedge \dots \wedge dx^{\mu_n} \epsilon_{\mu_1 \dots \mu_n}, \\ &= \det(e_{\mu}^a) dx^1 \wedge \dots \wedge dx^n. \end{aligned} \quad (2.92)$$

But,

$$\begin{aligned} \det(g) &= \det e^2 \\ \Rightarrow \det e &= \sqrt{\det g} \end{aligned} \quad (2.93)$$

Thus the required volume element corresponding to the metric $g_{\mu\nu}$ is

$$\sqrt{\det g} dx^1 \wedge \dots \wedge dx^n. \quad (2.94)$$

2.4.6 Interior product

The i_X -operator, known as the interior product maps a p -form to a $(p-1)$ -form. i.e. $i_X : i_X \omega^{(p)} \rightarrow \omega^{(p-1)}$, where X is a vector field given by $X = \sum_a f^a \frac{\partial}{\partial x^a}$. This is done by using the vector field X to eliminate one of the dx -factors present in the local representation of a p -form. We start with the local expression of a p -form:

$$\omega^{(p)} = \sum_{i_1, \dots, i_p} \omega_{i_1, \dots, i_p}(x) dx^{i_1} \wedge \dots \wedge dx^{i_p}. \quad (2.95)$$

$i_X \omega^{(p)}$ is then defined as

$$i_X \omega^{(p)} = \sum_{i_1, \dots, i_p, a} f^a(x) \omega_{i_1, \dots, i_p, a}(x) (-1)^a dx^{i_1} \wedge \dots \wedge \widehat{dx^a} \wedge \dots \wedge dx^{i_p}. \quad (2.96)$$

As always, the “caret” $\widehat{}$ over dx^a implies that the entry is missing. An immediate consequence of this definition is

$$\boxed{i_X^2 \omega^{(p)} = 0}, \quad (2.97)$$

since

$$\begin{aligned} i_X^2 \omega^{(p)} &= \sum_{i_1, \dots, i_p, a, b} f^a(x) f^b(x) \omega_{i_1, \dots, i_p, a, b} (-1)^a (-1)^b dx^{i_1} \wedge \dots \\ &\quad \wedge \widehat{dx^a} \wedge \dots \wedge \widehat{dx^b} \wedge \dots \wedge dx^{i_p}. \end{aligned} \quad (2.98)$$

As ω is anti-symmetric but $f^a(x)f^b(x)$ is symmetric, hence i_X^2 is zero.

Action of i_X -operator on wedge products

Again using the local expression for $\omega^{(p)}$ and $\mu^{(q)}$ we proceed to establish

$$\boxed{i_X(\omega^{(p)} \wedge \mu^{(q)}) = i_X \omega^{(p)} \wedge \mu^{(q)} + (-1)^p \omega^{(p)} \wedge i_X \mu^{(q)}} \quad (2.99)$$

To prove this we note

$$\begin{aligned} &i_X(\omega^{(p)} \wedge \mu^{(q)}) \\ &= i_X \left[\sum_{i_1, \dots, i_{p+q}} \omega_{i_1, \dots, i_p}(x) dx^{i_1} \wedge \dots \wedge dx^{i_p} \wedge \dots \wedge dx^{i_{p+q}} \mu_{i_{p+1}, \dots, i_{p+q}}(x) \right] \\ &= \sum_{i_1, \dots, i_{p+q}} [(-1)^a f^a(x) \omega_{i_1, \dots, i_p, a}(x) \mu_{i_{p+1}, \dots, i_{p+q}}(x) dx^{i_1} \wedge \dots \wedge \widehat{dx^a} \wedge \\ &\quad \dots \wedge dx^{i_p} \wedge dx^{i_{p+1}} \wedge \dots \wedge dx^{i_{p+q}} + (-1)^b \omega_{i_1, \dots, i_p}(x) \mu_{i_{p+1}, \dots, i_{p+q}}(x) \\ &\quad dx^{i_1} \wedge \dots \wedge dx^{i_p} \wedge dx^{i_{p+1}} \wedge \dots \wedge \widehat{dx^b} \wedge \dots \wedge dx^{i_{p+q}}] \end{aligned} \quad (2.100)$$

where $(-1)^b = (-1)^{p+c}$ where c is from i_{p+1} to i_{p+q} .

Hence (2.100) can be written as

$$\begin{aligned} &i_X \omega^{(p)} \wedge \mu^{(q)} + \sum_{i_1, \dots, i_{p+q}} (-1)^p \omega_{i_1, \dots, i_p}(x) \mu_{i_{p+1}, \dots, i_{p+q}}(x) (-1)^c \\ &\quad dx^{i_1} \wedge \dots \wedge dx^{i_p} \wedge dx^{i_{p+1}} \wedge \dots \wedge dx^{i_{p+q}}, \\ &= i_X \omega^{(p)} \wedge \mu^{(q)} + (-1)^p \omega^{(p)} \wedge i_X \mu^{(q)}. \end{aligned} \quad (2.101)$$

Finally we note that there is a simple coordinate independent definition for the action of interior product operator i_X on a p -form $\omega^{(p)}$, namely

$$(i_X \omega^{(p)})(X_1, \dots, X_{p-1}) = \omega^{(p)}(X, X_1, \dots, X_{p-1}) \quad (2.102)$$

2.4.7 Hodge star operator

The Hodge star operator maps a p -form to an $(n - p)$ -form where n is the dimension of the manifold. We recall that the space of p -forms in an n -dimensional manifold is a vector space $\Lambda^{(p)}$ of dimension ${}^nC_p$. To see this we note that the basis elements of a p -form are the wedge product of p different one forms present in the space. If the space is n dimensional there are n different one forms present namely $dx^1, \dots, dx^n$. The basis elements of the space of p -forms are simply p -forms of the form $dx^{i_1} \wedge \dots \wedge dx^{i_p}$. There are ${}^nC_p$ different linearly independent elements. Thus the dimension of $\Lambda^{(p)}$ is ${}^nC_p$. The space of $(n - p)$ -forms for the same manifold is a vector space $\Lambda^{(n-p)}$ of dimension ${}^nC_{n-p}$. Thus

$$\dim \Lambda^{(n-p)} = {}^nC_{n-p} = \dim \Lambda^{(p)}. \quad (2.103)$$

This suggests that a mapping between these spaces is possible. The basic geometrical idea is that using $\omega^{(p)}$ and taking its wedge product with $\omega^{(n-p)}$ gives a top form, which is one dimensional and can be interpreted as being proportional to local volume element. Thus the star product can be implicitly defined in a coordinate independent way as

$$\underbrace{\alpha^{(p)}}_{p\text{-form}} \wedge \underbrace{\star \beta^{(p)}}_{(n-p)\text{-form}} = (\alpha, \beta) \cdot \underbrace{\tau}_{\text{volume } n\text{-form}}, \quad (2.104)$$

where $(\alpha, \beta) = \sum_p \frac{1}{p!} \alpha_{i_1, \dots, i_p} \beta^{i_1, \dots, i_p}$ and $\tau = \sqrt{|g|} dx^1 \wedge \dots \wedge dx^n$. ($|g| = |\det g_{ij}|$) From this expression, we will determine the local action of the Hodge star operator. Note the explicit dependence on the metric through the appearance of the volume term τ . We can write τ using the anti-symmetric properties of the wedge product as

$$\begin{aligned} \tau &= \sum_{i_1, \dots, i_n} \frac{1}{n!} \sqrt{|g|} \epsilon_{i_1 \dots i_n} dx^{i_1} \wedge \dots \wedge dx^{i_n}, \\ &= \sum_{i_{p+1}, \dots, i_n} \frac{1}{(n-p)!} \sqrt{|g|} \epsilon_{i_1 \dots i_{p+1} \dots i_n} dx^1 \wedge \dots \wedge dx^p \wedge dx^{i_{p+1}} \wedge \dots \wedge dx^{i_n}. \end{aligned}$$

Note, that the factorial terms that are there because of symmetry, for instance there are $n!$ identical contributions corresponding to the $n!$ orderings that are possible in $dx^{i_1} \wedge \dots \wedge dx^{i_n}$. By choosing an ordering of the labels $i_1 < \dots < i_n$ introduces a restriction on the summation of terms. There is no longer the $n!$ symmetry property. Hence for summations over variable $i_1, \dots, i_n$ restricted by this ordering and there is no need to divide by $n!$. In this section we

will use this property to carry out calculations without symmetry. At the final step we can always remove restrictions on the summations present and reintroduce an appropriate factorial.

Remember that $\epsilon_{i_1 \dots i_n} \mathcal{L}_X \mathcal{L}_Y$ is completely anti-symmetric in $i_1 \dots i_n$. We will now discuss for the case $p = 1$, i.e.

$$\alpha^{(1)} = \alpha_{\mu_1} dx^{\mu_1} \quad (2.105)$$

$$\beta^{(1)} = \beta_{\mu_2} dx^{\mu_2}. \quad (2.106)$$

Let us choose $\alpha^{(1)} = \alpha_1 dx^1$, i.e. we have freezed μ_1 , in other words any component of $\alpha^{(1)}$ is zero except α_1 . We also know the definition of the inner product

$$\begin{aligned} (\alpha, \beta) &\equiv \alpha_{\mu} \beta^{\mu}, \\ &= \alpha_{\mu} \beta_{\nu} g^{\mu\nu}, \\ &= \alpha_1 \beta_{\nu} g^{1\nu}. \end{aligned} \quad (2.107)$$

Hence

$$\begin{aligned} (\alpha, \beta) \tau &= \alpha_1 \beta_{\nu} g^{1\nu} \sqrt{|g|} \epsilon_{1i_2 \dots i_n} dx^1 \wedge dx^{i_2} \wedge \dots \wedge dx^{i_n}, \\ &= \alpha_1 \beta_{\nu} \sqrt{|g|} \epsilon^{\nu}_{i_2 \dots i_n} dx^1 \wedge dx^{i_2} \wedge \dots \wedge dx^{i_n}. \end{aligned} \quad (2.108)$$

and the LHS of expression (2.104) gives

$$\begin{aligned} \alpha^{(1)} \wedge (\star \beta^{(1)}) &= (\alpha_1 dx^1) \wedge (\star \beta_{\nu} dx^{\nu}), \\ &= \alpha_1 \beta_{\nu} [dx^1 \wedge (\star dx^{\nu})]. \end{aligned} \quad (2.109)$$

Since Eqn. (2.108) and (2.109) are same, hence

$$\begin{aligned} \star dx^{\nu} &= \sqrt{|g|} \epsilon^{\nu}_{i_2 \dots i_n} dx^{i_2} \wedge \dots \wedge dx^{i_n}, \\ \star dx^{\mu} &= \sqrt{|g|} g^{\mu\nu} \epsilon_{\nu i_2 \dots i_n} dx^{i_2} \wedge \dots \wedge dx^{i_n}. \end{aligned} \quad (2.110)$$

By a similar analysis, it can be shown that

$$\begin{aligned} \star(dx^{i_1} \wedge \dots \wedge dx^{i_n}) &= \frac{\sqrt{|g|}}{(n-p)!} \epsilon^{i_1 \dots i_p}_{i_{p+1} \dots i_n} dx^{i_{p+1}} \wedge \dots \wedge dx^{i_n}, \\ &= \frac{\sqrt{|g|}}{(n-p)!} g^{i_1 j_1} \dots g^{i_p j_p} \epsilon_{j_1 \dots j_p j_{p+1} \dots j_n} dx^{j_1} \wedge \dots \wedge dx^{j_n}. \end{aligned} \quad (2.111)$$

In the last step we have used the summation convention, i.e. the repeated indices are summed over.

$$\star^2 = (-1)^{n(n-p)}$$

We have seen that for the Hodge star operator

$$\begin{aligned} \star \omega^{(p)} &\rightarrow \omega^{(n-p)}, \\ \star \omega^{(n-p)} &\rightarrow \omega^{(n-(n-p))} = \omega^{(p)}, \\ \text{i.e. } \star \star \omega^{(p)} &\rightarrow \omega^{(p)}. \end{aligned} \quad (2.112)$$

Explicitly

$$\star(dx^{i_1} \wedge \dots \wedge dx^{i_p}) = \frac{\sqrt{g}}{(n-p)!} \epsilon^{i_1 \dots i_p}_{i_{p+1} \dots i_n} dx^{i_{p+1}} \wedge \dots \wedge dx^{i_n}. \quad (2.113)$$

So,

$$\begin{aligned} \star \star (dx^{i_1} \wedge \dots \wedge dx^{i_p}) &= \frac{\sqrt{g}}{(n-p)!} \epsilon^{i_1 \dots i_p}_{i_{p+1} \dots i_n} \star (dx^{i_{p+1}} \wedge \dots \wedge dx^{i_n}), \\ &= \frac{\sqrt{g}}{(n-p)!} \epsilon^{i_1 \dots i_p}_{i_{p+1} \dots i_n} \frac{\sqrt{g}}{p!} \epsilon^{i_{p+1} \dots i_n}_{j_1 \dots j_p} \\ &\quad dx^{j_1} \wedge \dots \wedge dx^{j_p} \\ &= \frac{\sqrt{g}}{(n-p)!} \frac{\sqrt{g}}{p!} g^{i_1 k_1} \dots g^{i_p k_p} \epsilon_{k_1 \dots k_p i_{p+1} \dots i_n} \\ &\quad g^{i_{p+1} k_{p+1}} \dots g^{i_n k_n} \epsilon_{k_{p+1} \dots k_n j_1 \dots j_p} dx^{j_1} \wedge \dots \wedge dx^{j_p} \\ &= \frac{\sqrt{g}}{(n-p)!} \frac{\sqrt{g}}{p!} g^{i_1 k_1} \dots g^{i_p k_p} g^{i_{p+1} k_{p+1}} \dots g^{i_n k_n} \\ &\quad (-1)^{n(n-p)} \epsilon_{k_1 \dots k_p i_{p+1} \dots i_n} \epsilon_{j_1 \dots j_p k_{p+1} \dots k_n} \\ &\quad dx^{j_1} \wedge \dots \wedge dx^{j_p} \end{aligned} \quad (2.114)$$

Hence

$$\boxed{\star \star (dx^{i_1} \wedge \dots \wedge dx^{i_p}) = (-1)^{n(n-p)} dx^{i_1} \wedge \dots \wedge dx^{i_p}}. \quad (2.115)$$

♠ Example:

Consider 2-dimensional Euclidean space for which the metric is $g_{ij} = \delta_{ij}$ and $g^{ij} = \delta_{ij}$. Hence $\sqrt{|g|} = 1$.

Since $\epsilon_{12} = 1$, we have the following actions of the Hodge star operator on the bases of the space

$$\star 1 = dx^1 \wedge dx^2.$$

Similarly,

$$\star dx^1 = dx^2,$$

$$\star dx^2 = -dx^1,$$

and,

$$\star(dx^1 \wedge dx^2) = 1.$$

Natural Operators: Lie derivative and Laplacian

There are two natural operators which can be constructed from the operators d , i_X and $\star$, which map a p -form to a p -form. We will show that these operators are nothing but the Lie derivative and the Laplacian operator. First we have the operator

$$i_X d + di_X. \quad (2.116)$$

This operator depends linearly on the operator d . We will show that this operator is nothing but the Lie Derivative $\mathcal{L}_X$ (Cartan's formula). $\mathcal{L}_X = i_X d + di_X$

To do this, we simply show that

$$\mathcal{L}_X \omega^{(p)} = i_X d\omega^{(p)} + di_X \omega^{(p)}. \quad (2.117)$$

From the definition of i_X , it follows that

$$(i_X d\omega^{(p)})(X_1, \dots, X_p) = (d\omega^{(p)})(X, X_1, \dots, X_p),$$

we then use the expression derived earlier for $d\omega^{(p)}(X_1, \dots, X_p)$ to write

$$\begin{aligned} &= X(\omega^{(p)}(X_1, \dots, X_p)) + \sum_{j=1}^p (-1)^{j+2} \omega^{(p)}([X, X_j], \widehat{X}_1, \dots, \widehat{X}_j, \dots, X_p) \\ &\quad + \sum_{i=1}^p (-1)^{i+2} X_i(\omega^{(p)}(X, X_1, \dots, \widehat{X}_i, \dots, X_p)) \\ &\quad + \sum_{i < j} (-1)^{i+j+2} \omega^{(p)}([X_i, X_j], X, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_p). \end{aligned} \quad (2.118)$$

The first two terms represent $\mathcal{L}_X$, defined earlier in a coordinate free way, while the last two terms combined represent $-(di_X \omega^{(p)})(X_1, \dots, X_p)$. The sign factors take into account the presence of the extra vector field X .

Secondly we have the operator

$$d(\star d\star) + (\star d\star)d. \quad (2.119)$$

This again maps $\omega^{(p)} \rightarrow \omega^{(p)}$. Explicitly we have the sequence

$$\begin{aligned} d\omega^{(p)} &\rightarrow \omega^{(p+1)}, \\ \star d\omega^{(p)} &\rightarrow \omega^{n-(p+1)}, \\ d\star d\omega^{(p)} &\rightarrow \omega^{n-(p+1)+1}, \\ \star d\star d\omega^{(p)} &\rightarrow \omega^{n-[n-(p+1)+1]} = \omega^{(p)}. \end{aligned} \quad (2.120)$$

similarly one can check that

$$d\star d\star\omega^{(p)} \rightarrow \omega^{(p)} \quad (2.121)$$

We will now show that this operator acting on zero forms i.e. functions is nothing but the Laplacian operator. The operator introduced can thus be interpreted as the generalization of the Laplacian operator appropriate for acting on forms.

We separately evaluate each term of the expression

$$(d\star d\star + \star d\star d)f, \quad (2.122)$$

The first term gives zero. Since we know that $\star f$ should give us a n -form (since $\dim \mathcal{M} = n$), and hence $d\star f = 0$.

Now, let us look at the second term.

$$\begin{aligned} df &= \frac{\partial f}{\partial x^\mu} dx^\mu, \\ \star df &= \frac{\partial f}{\partial x^\mu} \star dx^\mu = \frac{\partial f}{\partial x^\mu} \sqrt{|g|} \epsilon^\mu_{i_2 \dots i_n} dx^{i_2} \wedge \dots \wedge dx^{i_n}, \\ &= \frac{\partial f}{\partial x^\mu} \sqrt{|g|} g^{\mu i_1} \epsilon_{i_1 i_2 \dots i_n} dx^{i_2} \wedge \dots \wedge dx^{i_n}, \end{aligned}$$

$$\begin{aligned}
d \star df &= \frac{\partial}{\partial x^\lambda} \left[\frac{\partial f}{\partial x^\mu} \sqrt{|g|} g^{\mu_1} \right] \epsilon_{i_1 \dots i_n} dx^\lambda \wedge dx^{i_2} \wedge \dots \wedge dx^{i_n}, \\
&= \frac{\partial}{\partial x^{i_1}} \left[\frac{\partial f}{\partial x^\mu} \sqrt{|g|} g^{\mu_1} \right] \epsilon_{i_1 \dots i_n} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_n}, \\
\star d \star df &= \frac{\partial}{\partial x^{i_1}} \left[\frac{\partial f}{\partial x^\mu} \sqrt{|g|} g^{\mu_1} \right] \epsilon_{i_1 \dots i_n} \star (dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_n}), \\
&= \frac{\partial}{\partial x^{i_1}} \left[\frac{\partial f}{\partial x^\mu} \sqrt{|g|} g^{\mu_1} \right] \epsilon_{i_1 \dots i_n} g^{i_1 j_1} \dots g^{i_n j_n} \epsilon_{j_1 \dots j_n} \sqrt{|g|}, \\
&= \frac{\partial}{\partial x^{i_1}} \left[\frac{\partial f}{\partial x^\mu} \sqrt{|g|} g^{\mu_1} \right] \underbrace{\epsilon_{i_1 \dots i_n} \epsilon^{i_1 \dots i_n}}_{\frac{1}{|g|}} \sqrt{|g|}, \\
&= \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^{i_1}} \left[\frac{\partial f}{\partial x^\mu} \sqrt{|g|} g^{\mu_1} \right]
\end{aligned} \tag{2.123}$$

We can denote $\star d \star$ by $d^\dagger$, then the Laplacian is

$$\nabla_p^2 \equiv \Delta_p = (d^\dagger d + d d^\dagger). \tag{2.124}$$

Chapter 3

Integration on manifolds

3.1 Integration on the Manifold $\mathcal{M}$

In this section we establish Stoke's Theorem. This states that the integral of a $(n - 1)$ -form $\omega^{(n-1)}$ over a $(n - 1)$ -dimensional boundary $\partial\mathcal{M}$ is equal to the integral of $d\omega^{(n-1)}$ over the n -dimensional manifold which has $\partial\mathcal{M}$ as its boundary.

$$\int_{\mathcal{M}} d\omega^{(n-1)} = \int_{\partial\mathcal{M}} \omega^{(n-1)} \quad (3.1)$$

In order to do this, we need to introduce two ideas. First we need to be able to represent a boundary in terms of Euclidean space and coordinates. Second, we need a way to reduce calculations to a sum over local coordinate charts calculations in a nice way.

We first introduce boundaries in Euclidean space. A standard method of doing this is to introduce a generalization of the idea of a upper half plane. We can motivate the procedure we follow by considering a two dimensional example. In two dimensions, we can map small regions into a square which is labelled by Euclidean variables x, y , where $-l < x < l$ and $-l < y < l$. The boundary region can be represented geometrically by choosing the boundary curve to be the x -axis. Now, x will again range between $-l$ and l , but y is restricted to lie between 0 and l (see figure). The generalization to higher dimensions (say n) replaces the square by a n -dimensional hypercube with cartesian coordinates $(x^1, \dots, x^n)$, all ranging between $-l$ and l , while the boundary is represented by coordinates $x^1, \dots, x^n$ in a region defined by $-l < x^1, \dots, x^{n-1} < l$ and $0 < x^n < l$. By a suitable choice of l such a collection of elements from an Euclidean space can be used to describe the given manifold. The second objective is achieved by requiring the manifold to be paracompact

and by introducing the idea of a partition of unity subordinate to the cover used to describe the manifold. This reduces the integration problems to the considerations of local coordinate charts.

3.1.1 Partition of unity

A partition of unity introduces a family of functions f_α satisfying the following conditions

1. f_α are smooth,
2. $f_\alpha \neq 0$ in U_α and is locally compact on U_α . Where U_α are locally finite cover of the manifold. This property is valid for a paracompact manifold. Local compactness means that each f_α vanishes outside and on the boundary of U_α while paracompactness of the manifold means that any given point of the manifold belongs to only a finite number of coordinate neighborhoods U_α .
3. $0 < f_\alpha < 1$,
4. $\sum_\alpha f_\alpha = 1$.

Now,

$$\begin{aligned}
 \int_{\mathcal{M}} d\omega^{(n-1)} &= \int_{\mathcal{M}} (d\omega^{(n-1)}) \sum_\alpha f_\alpha, \\
 &= \sum_\alpha \int_{\mathcal{M} \cap U_\alpha} f_\alpha d\omega^{(n-1)}, \\
 &= \sum_\alpha \int_{\mathcal{M} \cap U_\alpha} d(f_\alpha \omega^{(n-1)})
 \end{aligned} \tag{3.2}$$

where we have used the property of forms under maps in the last expression. We write

$$\omega^{(n-1)} = \sum_{j=1}^n (-1)^{j-1} h_j dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^n, \tag{3.3}$$

The manifold we are considering has a boundary $\partial\mathcal{M}$. Such a manifold can be covered by coordinate charts

$$U_\alpha : \{x^i(p) \mid |x^i(p)| < l\} \text{ in Euclidean space,} \tag{3.4}$$

For charts that do not include the boundary points the variables x^i for $i = 1, \dots, n$ are all in the range $-l \leq x^i \leq l$, while for the charts that include boundary points, the variables x^i have a different range, namely $-l \leq x^i \leq l$, for $i = 1, \dots, (n-1)$ and $0 \leq x^n \leq l$. Here boundary points correspond to $x^n = 0$.

Note that

$$d\omega^{(n-1)} = \sum_j (-1)^{j-1} \frac{\partial h_j}{\partial x^j} dx^j \wedge dx^1 \wedge \dots \wedge \widehat{dx^j} \wedge \dots \wedge dx^n. \quad (3.5)$$

Note also that when we move dx^j to the missing dx^j denoted by $\widehat{dx^j}$, $(-1)^{j-1}$ goes away. In view of the fact that the functions f_α from the partition of unity vanish outside U_α and on its boundary, we can consider contribution to Eqn. (3.2) from each coordinate chart U_α separately. We start by considering the general chart which includes boundary points.

$$\int_{U_\alpha} d\omega^{(n-1)} = \sum_j \int_{U_\alpha} \frac{\partial h_j}{\partial x^j} dx^1 \wedge \dots \wedge dx^j \wedge \dots \wedge dx^n, \quad (3.6)$$

Note that $f_\alpha = 0$, $x = \pm l$,

$$\int_{U_\alpha} d\omega^{(n-1)} = \sum_j \int_{-l}^l dx^1 \dots \int_{-l}^l \frac{\partial h_j}{\partial x^j} dx^j \dots \int_{-l}^l dx^n \quad (3.7)$$

There are two cases to consider

1. $\partial\mathcal{M} \cap U_\alpha = \emptyset$ i.e. $j \neq n$ Then

$$\begin{aligned} & \sum_j \int_{-l}^l dx^1 \dots \int_{-l}^l \frac{\partial h_j^\alpha}{\partial x^j} dx^j \dots \int_{-l}^l dx^n, \\ &= \sum_j \int_{-l}^l dx^1 \dots [h_j^\alpha(x_1, \dots, l, \dots, x_n) - h_j(x_1, \dots, -l, \dots, x_n)] \\ &= 0 \quad \text{since } f_\alpha \text{ vanishes for } x^i = \pm l \end{aligned} \quad (3.8)$$

2. $\partial\mathcal{M} \cap U_\alpha \neq \emptyset$ i.e. $j = n$,

$$\begin{aligned} & \int_{-l}^l dx^1 \dots \int_{-l}^l dx^{n-1} \int_0^l \frac{\partial h_n^\alpha}{\partial x^n} \\ &= \int_{-l}^l dx^1 \dots \int_{-l}^l dx^{n-1} [-h_n^\alpha(x_1, \dots, x_{n-1}, 0)], \end{aligned} \quad (3.9)$$

Where $h_j^\alpha = f_\alpha h_j$. We have used the property that $f_\alpha(x)$ vanishes when any one of its arguments is equal to $\pm l$, however f_α does not vanish when one of its arguments is zero. We claim that the expression given above is

$$\int_{\partial\mathcal{M}\cap U_\alpha} \omega^\alpha, \quad (3.10)$$

where $\omega^\alpha = \omega f_\alpha$. To establish this result we need to use the definition of ω and establish the volume element properly oriented for the boundary $\partial\mathcal{M}$.

We recall that

$$\omega^\alpha = \sum_j (-1)^{j-1} h_j^\alpha dx^1 \wedge \cdots \wedge dx^j \cdots \wedge dx^n. \quad (3.11)$$

On the boundary, $x^n = 0$ and $dx^n = 0$. Thus this expression collapses to

$$\omega^\alpha|_{\partial\mathcal{M}} = (-1)^{n-1} h_n^\alpha(x_1, \dots, x_{n-1}, 0) dx^1 \wedge \cdots \wedge dx^{n-1}. \quad (3.12)$$

modulo a sign factor which represents the proper orientation of $\partial\mathcal{M}$.

We now claim that the proper sign factor for the volume element on $\partial\mathcal{M}$ is

$$dx^1 \wedge \cdots \wedge dx^{n-1} (-1)^n \quad (3.13)$$

Taking this into account we get

$$\begin{aligned} & \int_{\partial\mathcal{M}} (-1)^{n-1} (-1)^n h_n(x_1, \dots, x_{n-1}, 0) dx^1 \wedge \cdots \wedge dx^{n-1} \\ &= \int_{\partial\mathcal{M}} \omega \end{aligned} \quad (3.14)$$

This orientation sign factor can be determined as follows: for a oriented manifold with basis vectors $e_1 \wedge \cdots \wedge e_n$, a positive orientation of the coordinates can be checked by evaluating $e_1 \wedge \cdots \wedge e_n$ on a set of coordinate directions given by the vectors $v_1, \dots, v_n$, and verifying that the sign of the determinant obtained is positive. Explicitly:

$$\begin{pmatrix} e_n(v_1) & e_{n-1}(v_1) & \cdots & e_1(v_1) \\ e_n(v_2) & e_{n-1}(v_2) & \cdots & e_1(v_2) \\ \cdots & \cdots & \cdots & \cdots \\ e_n(v_n) & e_{n-1}(v_n) & \cdots & e_1(v_n) \end{pmatrix} \quad (3.15)$$

The standard convention for fixing the orientation of a boundary is to choose “outward” directions for the normals to the boundary hypersurface. This

simply corresponds to reversing the sign of all v_n where v_n is the normal to the boundary. In the explicit formula given above this means changing the sign of the first column. This gives a factor $(-1)^n$. We choose this orientation sign factor. Summing over all f_α now establishes the theorem.

Stokes theorem can be written in the following suggestive way

$$\langle \mathcal{M}, d\omega \rangle = \langle \partial\mathcal{M}, \omega \rangle$$

where $\langle \mathcal{M}, d\omega \rangle = \int_{\mathcal{M}} d\omega$, $\langle \partial\mathcal{M}, \omega \rangle = \int_{\partial\mathcal{M}} \omega$. (3.16)

This way of stating the Stoke's theorem suggests that the d -operator which is a local differential operator is in some sense the "adjoint" of the geometrical non-local boundary.

Chapter 4

Applications

4.1 Formulation of Classical Hamiltonian Mechanics

The two fundamental theories of classical physics: mechanics and electromagnetic theory can be formulated in the language of differential forms. We briefly describe how this can be done. We start with the Hamiltonian formulation of mechanics. We will see that Hamiltonian Classical mechanics has a deep underlying structure called symplectic geometry. We proceed to uncover this structure. We start by considering an even dimensional manifold where a special type of two form ω is present. The two form has symmetry. The special nature of the two form ω introduced is that it is closed ie $d\omega = 0$ and is non degenerate ie $\omega(x, y) = 0$ for all vectors x implies $y = 0$. The even dimension nature of the space reflects the fact that in Hamiltonian mechanics both coordinates and momenta are required. The symmetry of the system requires ω to vanish under by the action of a Lie derivative.

An even dimensional manifold with such a globally define closed, non degenerate two form ω is called a symplectic manifold. The vector field, associated with the Lie derivative, is called a Hamiltonian vector field. To unravel the consequences of the symmetry required for ω we need to recall the Poincaré lemma: If locally $d\omega = 0$, then ω can be written as $\omega = d\eta$. Then ω is called locally exact. The final ingredient we need is symmetry. This is provided by a vector field. Let us choose a vector field, defining the Lie derivative, as follows

$$\mathcal{L}_V = V^\mu \frac{\partial}{\partial x^\mu}, \quad (4.1)$$

The symmetry we require is:

$$\mathcal{L}_V \omega = 0. \quad (4.2)$$

We have earlier shown that

$$\mathcal{L}_V \omega = i_V d + di_V. \quad (\text{Cartan's formula}) \quad (4.3)$$

Since ω is closed.

$$\mathcal{L}_V \omega = di_V \omega = 0, \quad [\text{since } i_V d\omega = 0], \quad (4.4)$$

Using Poincaré lemma, proved in chapter 2, we can write

$$i_V \omega = dH \quad (4.5)$$

where, H is a function (the Hamiltonian).

$$\begin{aligned} i_V \omega &= V^\mu \omega_{\mu\nu} = d_\nu H, \\ \text{or, } V^\mu &= \omega^{\mu\nu} d_\nu H, \\ \text{or, } V^\lambda &= \omega^{\mu\nu} \partial_\nu H \partial_\mu x^\lambda. \end{aligned} \quad (4.6)$$

Where, $\partial_\nu \Rightarrow \frac{\partial}{\partial x^\nu}$ and $\partial_\mu x^\lambda = \delta_\mu^\lambda$. Let us consider a simple two dimensional example where we choose $\omega = dx^1 \wedge dx^2$. Clearly $d\omega = 0$. The given choice for ω means that $\omega_{\mu\nu}$, the local coefficients defining ω , can be written in matrix form as

$$\omega_{\mu\nu} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{and } \omega^{\mu\nu} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (4.7)$$

Writing $x^1 = p$ and $x^2 = q$,

$$\begin{aligned} \dot{x}^1 &= \frac{dx^1}{dt} = \omega^{12} \partial_2 H \partial_1 x^1 + \omega^{21} \partial_1 H \partial_2 x^1, \\ \text{thus, } \dot{x}^1 &= -\frac{\partial H}{\partial x^2}, \\ \text{i.e., } \dot{p} &= -\frac{\partial H}{\partial q}. \end{aligned} \quad (4.8)$$

Similarly,

$$\begin{aligned} \dot{x}^2 &= \frac{dx^2}{dt} = \omega^{12} \partial_2 H \partial_1 x^2 + \omega^{21} \partial_1 H \partial_2 x^2, \\ \Rightarrow \dot{q} &= \frac{\partial H}{\partial p} \end{aligned} \quad (4.9)$$

These are the standard Hamilton's Equations with q as coordinate and p as the momentum.

For symplectic geometry, we have the remarkable result

◇ **Darboux' Theorem:**

Let ω be a closed two form on a $2p$ dimensional manifold $\mathcal{M}$, then about every point of $\mathcal{M}$ we can introduce coordinates $x^1, \dots, x^p, y^1, \dots, y^p$, such that in terms of these coordinates

$$\omega = dx^1 \wedge dy^1 + \dots + dx^p \wedge dy^p \quad (4.10)$$

This theorem means that the local symplectic forms on arbitrary manifolds are the same irrespective of the manifold. This is very different from Riemannian geometry, where the metric tensor cannot always be chosen to make the space Euclidean.

4.2 Maxwell's Equations

Maxwell's equations can be written in terms of differential forms. We illustrate this by considering a system without charges or currents. The claim is that Maxwell's equations are then given by

$$dF = 0, \quad (4.11)$$

$$\text{and } d \star F = 0. \quad (4.12)$$

Where,

$$\begin{aligned} F &= F_{\mu\nu} dx^\mu \wedge dx^\nu, \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \end{aligned} \quad (4.13)$$

Here the electric field

$$E_i = F_{0i} = \partial_0 A_i - \partial_i A_0, \quad (4.14)$$

and the magnetic field

$$F_{ij} = \partial_i A_j - \partial_j A_i = \epsilon_{ijk} B_k. \quad (4.15)$$

Using these identifications we can write

$$\begin{aligned} F &= dt \wedge (dx^1 E_1 + dx^2 E_2 + dx^3 E_3) \\ &\quad - dx^1 \wedge dx^2 B_3 - dx^3 \wedge dx^1 B_2 - dx^2 \wedge dx^3 B_1. \end{aligned}$$

Calculating $dF = 0$ gives the equations

$$\begin{aligned}\vec{\nabla} \times \vec{E} &= -\vec{B}, \\ \vec{\nabla} \cdot \vec{B} &= 0.\end{aligned}\tag{4.16}$$

Calculating $d \star F = 0$ gives the equations

$$\begin{aligned}\vec{\nabla} \cdot \vec{E} &= 0, \\ \vec{\nabla} \times \vec{B} &= \vec{E}.\end{aligned}\tag{4.17}$$

4.3 Frobenius' Theorem

Frobenius' theorem is used to answer two different types of questions. One has to do with differential equations. Given a set of ordinary differential equations which are over-determined i.e. there are more equations than unknowns, is there a solution? The second question it helps answer is geometrical, namely, does a p -dimensional submanifold $\mathcal{N}$ of an n -dimensional manifold $\mathcal{M}$ exist with prescribed tangent planes? For the one dimensional case, the geometrical question has a positive answer. We have seen earlier that a curve with prescribed tangent vectors can be constructed locally. For the more general situation described the answer in general is negative. If a p -dimensional submanifold with a specific set of tangent planes exist, then the submanifold $\mathcal{N}$ is said to be integrable.

To state the theorem we need to introduce p -vector fields $X_1, \dots, X_p$ on the submanifold and their associated p dual differential one forms. The geometrical property which is crucial is the statement that the commutator of any two of the vectors from this set can be expressed as a linear combination of the vectors $X_1, \dots, X_p$.

In a neighborhood of a point c in $\mathcal{N}$, we choose

$$X_i = \frac{\partial}{\partial x^i}, \quad i = 1, \dots, p.\tag{4.18}$$

as basis vectors. These generate a dual basis of one forms

$$(dx^1, \dots, dx^p)\tag{4.19}$$

on the submanifold $\mathcal{N}$. One version of Frobenius' theorem states that $\mathcal{N}$ is locally integrable if general vector fields i.e. $X_i = a_i(x) \frac{\partial}{\partial x^i}$, $i = 1, \dots, p$, defined on $\mathcal{N}$ satisfy

$$[X_i, X_j] = c_{ij}^k X_k, \quad i, j, k = 1, \dots, p,\tag{4.20}$$

There is an equivalent statement of Frobenius' theorem in terms of differential one forms which states that $\mathcal{N}$ is integrable if

$$d\omega_\alpha = \sum_{\beta=p+1}^n \Omega_{\alpha\beta} \wedge \omega_\beta, \quad \alpha = p+1, \dots, n.$$

and the set of one forms ω^α vanishes on the submanifold

These equations are known as Pfaffian differential equations. This could be an over complete set of equations valid on the submanifold. On the submanifold $\mathcal{N}$, dx^i , $i = 1, \dots, p$ are basis vectors. Restricting to $\mathcal{N}$

$$dx^\alpha = \sum_{i=1}^p c_i^\alpha dx^i, \quad \alpha = p+1, \dots, n, \quad (4.21)$$

since dx^i , ($i = 1, \dots, p$) form a basis in $\mathcal{N}$.

define one forms

$$w^\alpha = dx^\alpha - \sum_{i=1}^p c_i^\alpha dx^i, \quad (4.22)$$

These vanish on $\mathcal{N}$ by construction however away from $\mathcal{N}$, w_α need not be zero. If we can show that $w^\alpha = df^\alpha = 0$ on U_α (a coordinate neighbourhood of $\mathcal{N}$), then it follows that $f^\alpha = (\text{constant})_\alpha$. These equations define coordinate system for the submanifold $\mathcal{N}$ with prescribed tangent planes as we will show. The tangent planes are defined by vectors which satisfy the commutation property stated. We will use

$$dw^\alpha(X_i, X_j) = \sum_{i=1}^p (-)^{i+j} X_i w^\alpha(X_j) - \sum_{i < j, j=1, \dots, p} w^\alpha([X_i, X_j]) \quad (4.23)$$

Where $w^\alpha = \sum_i w_i^\alpha dx^i$. Locally, on U_α ,

$$\begin{aligned} w^\alpha &= \sum_{i=1}^p w_i^\alpha dx^i \\ w^\alpha|_{U_\alpha} = 0 &\Rightarrow w_i^\alpha|_{U_\alpha} = 0, \text{ where } w_i^\alpha = w^\alpha(X_i). \end{aligned} \quad (4.24)$$

Also,

$$\begin{aligned} w^\alpha([X_i, X_j]) &= w^\alpha[c_{ij}^k X_k] - c_{ij}^k w^\alpha(X_k) \\ &= c_{ij}^k w_k^\alpha \\ &= 0 \text{ on } U_\alpha. \end{aligned} \quad (4.25)$$

Hence $dw^\alpha(X_i, X_j) = 0$, which implies by using Poincaré lemma that $w^\alpha = df^\alpha$, but w^α also vanishes on U_α , hence the result follows.

We have established the sufficiency of the theorem. The necessary part is straightforward and is left as an exercise

4.3.1 Two dimensional surface in three dimensions

We study the conditions needed for a two dimensional surface with prescribed three dimensional tangent planes to exist locally. We suppose that the tangent planes are defined by two vector fields

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x} + f(x, y) \frac{\partial}{\partial z}, \\ X_2 &= \frac{\partial}{\partial y} + g(x, y) \frac{\partial}{\partial z}, \end{aligned} \quad (4.26)$$

where f, g are arbitrary functions of (x, y) . We want to find out if there is a subspace which has X_1, X_2 as its tangents. Let $(x, y, z(x, y))$ are the coordinates in $\mathcal{N}$. The tangent vectors

$$\hat{X}_1 = \frac{\partial}{\partial x} + \frac{\partial z}{\partial x} \frac{\partial}{\partial z}, \quad (4.27)$$

$$\hat{X}_2 = \frac{\partial}{\partial y} + \frac{\partial z}{\partial y} \frac{\partial}{\partial z}. \quad (4.28)$$

Locally we require $\hat{X}_1 = X_1$ and $\hat{X}_2 = X_2$, which means

$$\begin{aligned} \frac{\partial z}{\partial x} &= f, \\ \frac{\partial z}{\partial y} &= g. \end{aligned}$$

Consistency of these equations gives the condition

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} \quad (4.29)$$

It is clear that for arbitrary f and g will not be satisfied. Thus $\mathcal{N}$ would not exist. Let us see how this confusion comes from a direct application of Frobenius' theorem. From the theorem, it follows that $\mathcal{N}$ exists only if the commutator $[X_1, X_2]$ is linearly related to X_1 and X_2 . We find

$$\begin{aligned} [X_1, X_2] &= \left[\frac{\partial}{\partial x} + f \frac{\partial}{\partial z}, \frac{\partial}{\partial y} + g \frac{\partial}{\partial z} \right], \\ &= \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y} + g \frac{\partial}{\partial z} \right] + \left[f \frac{\partial}{\partial z}, \frac{\partial}{\partial y} + g \frac{\partial}{\partial z} \right], \\ &= \left[\frac{\partial}{\partial x}, g \right] \frac{\partial}{\partial z} + \left[f \frac{\partial}{\partial z}, \frac{\partial}{\partial y} \right], \\ &= \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \frac{\partial}{\partial z}. \end{aligned}$$

The right hand side is clearly not a linear combination of X_1 and X_2 , since it involves derivative of f and g . The only condition for consistency for the commutator to vanish. This gives the same result as the direct geometrical approach.

References for further reading

A short list of references for further reading

- Differentiable Manifolds, *Yozo Matsushima*, Marcel Dekker, Inc. : An excellent text with proofs and examples.
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